BEST STUDENT EXAM OPEN
Texas A&M High School Math Contest
November 12, 2022

Directions: *Answers should be simplified, and if units are involved include them in your answer.*

Problem 1. What is the sum of all positive integers n such that $6n$ is divisible by $1 + 2 + \cdots + n$?

Problem 2. It is given that one root of $2x^2 + rx + s = 0$, with r and s real numbers, is $3+2i$ ($i = \sqrt{-1}$). What is the value of $r + s$?

Problem 3. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the difference between the greatest and the least possible perimeters of the triangle?

Problem 4. A family consists of a mother, a father, and several children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

Problem 5. The number $25^{64} \cdot 64^{25}$ is the square of a positive integer N. What is the sum of the digits in the decimal representation of N?

Problem 6. A five-digit number is of the form $\overline{bbcac}$, where $0 \le a < b < c \le 9$, and b is the average of a and c. How many different five-digit numbers satisfy all these properties?

Problem 7. Evaluate

$$\lim_{n\to\infty} \cos^n\left(\sqrt{\frac{2022}{n}}\right).$$

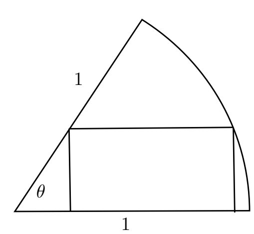
Problem 8. Find $\sin\theta + \cos\theta$ if we know that $\sin^3\theta + \cos^3\theta = \frac{11}{16}$.

Problem 9. Evaluate

$$\int_1^{2022} \frac{\{x\}^{\lfloor x \rfloor}}{\lfloor x \rfloor}\, dx$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Problem 10. A rectangle is inscribed in a sector of a circle of radius 1 as shown in the figure. The central angle of the sector is $\theta = \pi/3$. What is the maximum possible area for the rectangle?



Problem 11. Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{10^n}.$$

Problem 12. How many positive perfect squares have five or fewer digits, and have a 1, 2, or 3 as their leftmost digit?

Problem 13. In the triangle ΔABC points D, E, and F divide the segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, in the ratio 2:1, i.e., $BD/DC = CE/EA = AF/FB = 2$. What is the ratio of the area of the triangle formed by $\overline{AD}, \overline{BE}$, and $\overline{CF}$ to the area of the triangle ΔABC?

Problem 14. Evaluate

$$\int_{\alpha}^{\beta} \cos\left(x - \frac{1}{x}\right) dx,$$

where $\alpha = \frac{1}{6}\left(\sqrt{36+\pi^2} - \pi\right)$, $\beta = \frac{1}{6}\left(\sqrt{36+\pi^2} + \pi\right)$.

Problem 15. What is the prime factorization of $1,003,003,001$? Write your answer in the increasing order of prime bases abbreviating repeated factors by the use of exponents (powers).

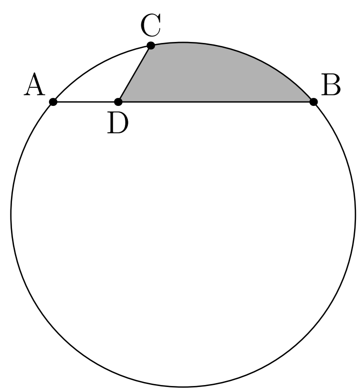
Problem 16. Suppose you repeatedly toss a fair coin until you get two heads in a row. What is the probability that you stop on the 10th toss? Express your answer in reduced form.

Problem 17. Let $\mathbb{Z}$ be the set of all integer numbers. Suppose a function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies the identities

$$f(0) = 1, \qquad f(2a) + 2f(b) = f(f(a+b)) \quad \text{for all } a, b \in \mathbb{Z}.$$

What is $f(2022)$?

Problem 18. In the figure below, $AD = CD = 1$, $BD = 3$, and $\angle BDC = 60^\circ$. Compute the shaded area.



Problem 19. Monika has four distinct integers on her list. If she removes any integer from the list, the three remaining integers add up to a perfect square. What is the smallest possible value of Monika's greatest integer?

Problem 20. Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{n!\left(n - \frac{3}{4}\right)}{(2n)!}.$$