BEST STUDENT EXAM SOLUTIONS
Texas A&M High School Math Contest
November 12, 2022

Solution 1. Since we have $1 + 2 + \cdots + n = n(n+1)/2$, the condition occurs if the ratio

$$\frac{6n}{\frac{n(n+1)}{2}} = \frac{12}{n+1}$$

is an integer, meaning that $n+1$ must be one of the positive factors of 12, so $n = 1, 2, 3, 5$, and 11, whose sum is $\boxed{22}$. ■

Solution 2. The other root must be $3 - 2i$. Therefore,

$$-\frac{r}{2} = (3+2i) + (3-2i) = 6, \qquad \frac{s}{2} = (3+2i)(3-2i) = 9 + 4 = 13.$$

Consequently, $r + s = -12 + 26 = \boxed{14}$.■

Solution 3. Let the sides of the triangle have lengths $x, 3x$, and 15. The Triangle Inequality implies that $3x < x + 15$, so $x < 7.5$. Because x is an integer, the greatest possible perimeter occurs when $x = 7$. Thus the greatest possible perimeter is $7 + 21 + 15 = 43$. On the other hand, the Triangle Inequality also implies that $x + 3x > 15$, so $x > 3.75$. Because x is an integer, the least possible perimeter occurs when $x = 4$, whence the least possible perimeter is $4 + 12 + 15 = 31$. The difference between the greatest and least possible perimeters, therefore, equals $43 - 31 = \boxed{12}$. ■

Solution 4. Let N represent the number of children in the family and T represent the sum of the ages of all the family members. The average age of the members of the family is 20, and the average age of the members when the 48-year-old father is not included is 16, so

$$20 = \frac{T}{N+2}, \qquad 16 = \frac{T-48}{N+1}.$$

This implies that

$$20N + 40 = T, \qquad 16N + 16 = T - 48,$$

so

$$20N + 40 = 16N + 64.$$

Hence $4N = 24$ and $N = \boxed{6}$.■

Solution 5. We have

$$N = \sqrt{(5^2)^{64} \cdot (2^6)^{25}} = 5^{64} \cdot 2^{3 \cdot 25} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048,$$

so the sum of the digits of N is $2 + 4 + 8 = \boxed{14}$. ■

Solution 6. Once a and c are chosen, the integer b is determined. For $a = 0$, we could have $c = 2, 4, 6$, or 8. For $a = 2$, we could have $c = 4, 6$, or 8. For $a = 4$, we could have $c = 6$ or 8, and for $a = 6$ the only possibility is $c = 8$. Thus there are $1 + 2 + 3 + 4 = 10$ possibilities when a is even. Similarly, there are 10 possibilities when a is odd, so the number of possibilities is $\boxed{20}$. ■

Solution 7. Notice that the limit is an indeterminate form of the type 1^∞. Let L be the limit we are trying to evaluate. Then we have, using the substitution $x = \sqrt{\frac{2022}{n}}$

$$\ln L = \lim_{n\to\infty} n \ln\left(\cos\left(\sqrt{\frac{2022}{n}}\right)\right) = \lim_{x\to 0} \frac{\ln(\cos(x))}{\left(\frac{x^2}{2022}\right)},$$

which is an indeterminate form of the type $\frac{0}{0}$. Applying the l'Hoptial's rule twice, we get

$$\ln L = \lim_{x\to 0} \frac{-\tan(x)}{\left(\frac{x}{1011}\right)} = \lim_{x\to 0} \frac{-\sec^2(x)}{\frac{1}{1011}} = -1011,$$

meaning that $L = \boxed{1/e^{1011}}$.

Solution 8. Let $x = \sin\theta + \cos\theta$. We know

$$x^2 = 1 - 2\sin\theta\cos\theta, \qquad \sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$$

Eliminating $\sin\theta\cos\theta$ we get the following cubic equation in x

$$\frac{11}{16} = x\left(1 - \frac{x^2 - 1}{2}\right),$$

which is reduced to

$$8x^3 - 24x + 11 = 0.$$

Trying to factor this polynomial, we first search for any rational root of the form p/q, where p and q are co-prime integers. By the Rational Root Theorem, p would be an integer factor of 11, so $p = \pm 1, \pm 11$ and q would be an integer factor of 4, so $q = \pm 1, \pm 2, \pm 4$. All choices fail, upon testing except for $x = 1/2$, meaning that $2x - 1$ is a factor of our cubic polynomial, and we can establish a factorization of the form

$$8x^3 - 24x + 11 = (2x - 1)(4x^2 + 2x - 11)$$

by long division. This results in three roots of the form

$$x = \frac{1}{2}, \qquad x = \frac{-1 \pm 3\sqrt{5}}{4},$$

among which the first one is the only acceptable choice due to the inequality $|\sin\theta + \cos\theta| \leq \sqrt{2}$. Indeed, as we have

$$\left|\frac{-1 + 3\sqrt{5}}{4}\right| < \left|\frac{-1 - 3\sqrt{5}}{4}\right|,$$

it suffices to show that $\frac{-1+3\sqrt{5}}{4} > \sqrt{2}$, which easily follows from the following argument:

$$\left(\frac{-1 + 3\sqrt{5}}{4}\right)^2 = \frac{46 - 6\sqrt{5}}{16} = \frac{23 - 3\sqrt{5}}{8} > \frac{23 - 7}{8} = 2 = (\sqrt{2})^2,$$

where we have used the fact that $3\sqrt{5} < 7$ as $(3\sqrt{5})^2 = 45 < 49 = 7^2$. Therefore, the answer is $\boxed{\frac{1}{2}}$. ■

Solution 9. Write this integral as a sum:

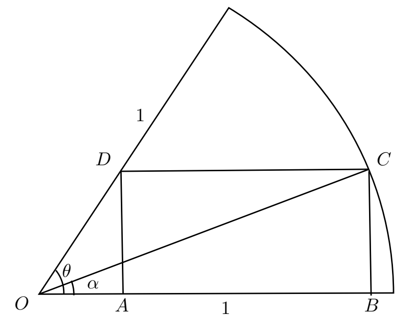
$$\sum_{n=1}^{2021} \int_n^{n+1} \frac{\{x\}^{\lfloor x \rfloor}}{\lfloor x \rfloor}\, dx = \sum_{n=1}^{2021} \int_n^{n+1} \frac{(x-n)^n}{n}\, dx = \sum_{n=1}^{2021} \int_0^1 \frac{x^n}{n}\, dx = \sum_{n=1}^{2021} \frac{1}{n(n+1)}$$

This telescopes as

$$\sum_{n=1}^{2021} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{2022} = \boxed{\frac{2021}{2022}}$$

■

Solution 10. Let $\alpha = \angle COB$ be the variable angle between 0 and θ. Since $OA = AD \cot\theta$ and $AD = BC = OC \sin\alpha = \sin\alpha$, we have



$$AB = OB - OA = OC\cos\alpha - AD\cot\theta = \cos\alpha - \sin\alpha\cot\theta,$$

hence the area $f(\alpha)$ of the rectangle $ABCD$ is

$$f(\alpha) = BC \cdot AB = \sin\alpha\cos\alpha - \sin^2\alpha\cot\theta = \frac{1}{2}\sin(2\alpha) - \sin^2\alpha\cot\theta.$$

To maximize the area, we find the derivative of $f(\alpha)$ as follows

$$f'(\alpha) = \cos(2\alpha) - \sin(2\alpha)\cot\theta,$$

so $f'(\alpha) = 0$ implies that $\cot(2\alpha) = \cot\theta$, namely, $\alpha = \theta/2 = \pi/6$. Therefore, the maximum area is

$$\frac{1}{2}\sin\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{12} = \boxed{\frac{\sqrt{3}}{6}} = \boxed{\frac{1}{2\sqrt{3}}}.$$

■

Solution 11. Let S be the sum. Then we can reformulate the sum as a double sum and evaluate it as follows.

$$S = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{10^n} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{10^n} = \sum_{k=1}^{\infty} \frac{\frac{1}{10^k}}{1-\frac{1}{10}} = \frac{10}{9}\cdot\sum_{k=1}^{\infty}\frac{1}{10^k} = \frac{10}{9}\cdot\frac{\frac{1}{10}}{1-\frac{1}{10}} = \boxed{\frac{10}{81}}.$$

Solution 12. The number of n-digit perfect squares whose leftmost digit is a 1, 2, or 3 is just the number of perfect squares in $[10^{n-1}, 4\times 10^{n-1})$, and this is

$$\left\lceil \sqrt{4\times 10^{n-1}} \right\rceil - \left\lceil \sqrt{10^{n-1}} \right\rceil$$

If n is odd, both square roots are integers so we have

$$2\cdot 10^{\frac{n-1}{2}} - 10^{\frac{n-1}{2}} = 10^{\frac{n-1}{2}}$$

So for $n = 1, 3, 5$ we have $1, 10, 100$ possible solutions respectively. If n is even, we have

$$\left\lceil 2\cdot 10^{\frac{n}{2}-1}\sqrt{10} \right\rceil - \left\lceil 10^{\frac{n}{2}-1}\sqrt{10} \right\rceil$$

So for $n = 2$ we have

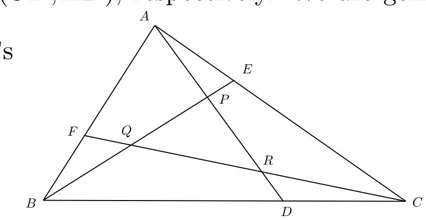
$$\left\lceil 2\sqrt{10} \right\rceil - \left\lceil \sqrt{10} \right\rceil = 7 - 4 = 3$$

and for $n = 4$ we have

$$\left\lceil 20\sqrt{10} \right\rceil - \left\lceil 10\sqrt{10} \right\rceil = 64 - 32 = 32$$

(We can either use crude approximations like $\sqrt{10} \approx 3.16$, or just use trial and error; e.g. for four-digit squares it's not too tedious to compute that $32^2 = 1024$ is the smallest and $63^2 = 3969$ is the largest.) Thus, summing these up gives $1 + 3 + 10 + 32 + 100 = \boxed{146}$. ■

Solution 13. Let P, Q, R be the intersection of the pairs $(\overline{AD}, \overline{BE}), (\overline{BE}, \overline{CF}), (\overline{CF}, \overline{AD})$, respectively. We are going to find the ratio between the areas of the triangles ΔPQR and ΔABC. Using Menelaus's theorem, for the triangle ΔABD and the transversal $\overline{CF}$, we have



$$\frac{AF}{FB} \times \frac{BC}{CD} \times \frac{DR}{RA} = 1,$$

hence $DR/RA = (\frac{AF}{FB} \times \frac{BC}{CD})^{-1} = (2 \times 3)^{-1} = 1/6$.

Another way to get this ratio is to use the method of center of mass (a.k.a mass point geometry): Place weights 2 lb, 1 lb, and 1/2 lb at points C, B, and A, respectively. Then D is the center of mass of the two points B and C, and R is the center of mass of the three points A, B, and C. Assigning the sum of the weights at B and C, i.e. 3, to their center of mass, D, we get that R is the center of mass of the two points D and A. Therefore,

$$\frac{DR}{RA} = \frac{(1/2)}{3} = \frac{1}{6}.$$

As a consequence,

$$\frac{\text{area}(\Delta ARC)}{\text{area}(\Delta ABC)} = \frac{\text{area}(\Delta ARC)}{\text{area}(\Delta ADC)} \times \frac{\text{area}(\Delta ADC)}{\text{area}(\Delta ABC)} = \frac{6}{7} \times \frac{1}{3} = \frac{2}{7}.$$

By symmetry, we have

$$\frac{\text{area}(\Delta ARC)}{\text{area}(\Delta ABC)} = \frac{\text{area}(\Delta CQB)}{\text{area}(\Delta ABC)} = \frac{\text{area}(\Delta BPA)}{\text{area}(\Delta ABC)} = \frac{2}{7},$$

hence

$$\frac{\text{area}(\Delta PQR)}{\text{area}(\Delta ABC)} = 1 - \left(\frac{\text{area}(\Delta ARC)}{\text{area}(\Delta ABC)} + \frac{\text{area}(\Delta CQB)}{\text{area}(\Delta ABC)} + \frac{\text{area}(\Delta BPA)}{\text{area}(\Delta ABC)}\right) = 1 - \left(\frac{2}{7} + \frac{2}{7} + \frac{2}{7}\right) = \boxed{\frac{1}{7}}.$$

■

Solution 14. First, notice that $\beta - \alpha = \frac{\pi}{3}$ and $\alpha\beta = 1$. Let I be the definite integral we are evaluating here. With the substitution $u = \frac{1}{x}$ we can easily convert the integral to the following

$$I = \int_\alpha^\beta \frac{1}{x^2} \cos\left(x - \frac{1}{x}\right) dx.$$

In particular,

$$I = \frac{1}{2}\left(\int_\alpha^\beta \cos\left(x - \frac{1}{x}\right) dx + \int_\alpha^\beta \frac{1}{x^2} \cos\left(x - \frac{1}{x}\right) dx\right) = \frac{1}{2}\int_\alpha^\beta \left(1 + \frac{1}{x^2}\right) \cos\left(x - \frac{1}{x}\right) dx,$$

which can be converted to the following, using the substitution $\theta = x - \frac{1}{x}$:

$$I = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta \, d\theta = \sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}.$$

■

Solution 15. Let N be the given number in the problem. Noticing the appearance of the binomial coefficients

$$\binom{3}{0} = 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1$$

in the decimal expansion of N implies the equality

$$N = \binom{3}{0}10^9 + \binom{3}{1}10^6 + \binom{3}{2}10^3 + \binom{3}{3}.$$

By binomial theorem we have

$$N = (10^3 + 1)^3 = (1001)^3.$$

The prime factorization of 1001 is $7 \cdot 11 \cdot 13$, so $N = \boxed{7^3 \cdot 11^3 \cdot 13^3}$. ■

Solution 16. There are 2^{10} possible outcomes when one flips a coin 10 times. Let T_i be the result of the i-th flip. We want to compute the number of possible outcomes which have $T_9 = T_{10} = H$ and no adjacent H's in $T_1, T_2, \dots, T_8, T_9$. Certainly then, $T_8 = T$ and we wish to know how many sequences of length 7 have no adjacent H's. Let S_n be the number of sequences of length n without 2 consecutive occurrences of H. Each such sequence must either start with HT followed by a sequence of length $n-2$ with no consecutive H's or start with T followed by a sequence of length $n-1$ with no consecutive H's. Therefore $S_n = S_{n-1} + S_{n-2}$. Also, $S_1 = 2, S_2 = 3$. We see then that $S_n = F_{n+2}$ where F_i denotes the i-th Fibonacci number. Hence $S_7 = F_9 = 34$. It follows that there are 34 possible outcomes which satisfy our condition. The probability that such an outcome occurs is then $34/1024 = \boxed{17/512}$.■

Solution 17. If we plug in $a = 0$ we get

$$1 + 2f(b) = f(f(b)), \quad \forall b \in \mathbb{Z}.$$

If we plug in $a = 1$ we get

$$f(2) + 2f(b) = f(f(b+1)),$$

Combining the two identities, we have

$$f(2) + 2f(b) = f(f(b+1)) = 1 + 2f(b+1), \quad \forall b \in \mathbb{Z}.$$

Consequently,

$$f(b+1) - f(b) = \frac{f(2) - 1}{2}, \quad \forall b \in \mathbb{Z},$$

which means that f must be an arithmetic progression 'starting' from 1, namely, $f(x) = mx+1$, for some $m \in \mathbb{Z}$. Substituting this form into the initial functional identity, we get

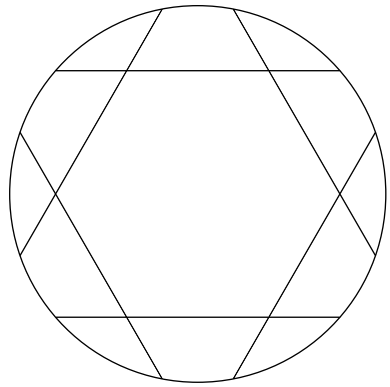
$$(m(2a) + 1) + 2(mb + 1) = m(m(a+b) + 1) + 1, \quad \forall a, b \in \mathbb{Z},$$

hence

$$(m^2 - 2m)(a+b) + m - 2 = 0, \quad \forall a, b \in \mathbb{Z}.$$

Since a, b are arbitrary, we must have $m = 2$, so $f(x) = 2x + 1, \quad \forall x \in \mathbb{Z}$ is the unique function with such properties. In particular, $f(2022) = \boxed{4045}$.■

Solution 18. We can draw six copies of chord AB, each one rotated 60 degrees. This will give the following picture:



where each chord is divided into segments of length 1, 2, and 1. This subdivides the circle into six copies of the region $\mathcal{R}$ we wish to find the area of, and one regular hexagon of side length 2 which has area $6\sqrt{3}$. We can easily find the radius of the circle too; since the center of the hexagon is also the center of the circle, we can drop an altitude to one of the sides (which has length $\sqrt{3}$). Then since the altitude bisects the chord, which has length 4, the radius is simply $\sqrt{3+2^2} = \sqrt{7}$ and so the circle has area 7π. Hence, the area of the region $\mathcal{R}$ is $\frac{7\pi - 6\sqrt{3}}{6} = \boxed{\frac{7\pi}{6} - \sqrt{3}}$. ■

Solution 19. Let a^2, b^2, c^2, d^2 be the four squares that the groups of three numbers sum to. Note that these squares are all distinct – if some two are the same, then that implies that some two numbers on Monika's list must be the same, but they are given to be distinct. So WLOG let $a^2 < b^2 < c^2 < d^2$. Then Monika's numbers are the solutions to the system

$$x + y + z = a^2$$

$$x + y + w = b^2$$

$$x + z + w = c^2$$

$$y + z + w = d^2$$

This system is easily solvable; for example, adding any three equations, then subtracting the fourth one twice, will give a solution for one of the variables. Explicitly, we have

$$x = K - d^2$$

$$y = K - c^2$$

$$z = K - b^2$$

$$w = K - a^2$$

where

$$K = \frac{a^2 + b^2 + c^2 + d^2}{3}$$

In particular, the greatest integer w in the list satisfies

$$w = K - a^2 = \frac{b^2 + c^2 + d^2 - 2a^2}{3} \tag{1}$$

Since $(n + 1^2 - n^2 = 2n + 1$ is strictly increasing with respect to $n \geq 0$, we have that $c^2 - a^2 \geq 2^2 - 0^2 = 4$ and $d^2 - a^2 \geq 3^2 - 0^2 = 9$. Besides $b^2 \geq 1$. Therefore $w \geq \frac{1+4+9}{3} = \frac{14}{3} > 4$. It also shows that when $a^2 = 0, b^2 = 1, c^2 = 4, d^2 = 9$, we get that $w = \frac{14}{3}$ is not integer and therefore $d^2 \geq 16$. Consequently, $d^2 - a^2 \geq 4^2 - 0^2 = 16$ and $w \geq \frac{1+4+16}{3} = \frac{21}{3} = 7$.

Moreover, the equality is achieved for $a^2 = 0, b^2 = 1, c^2 = 4, d^2 = 16$ and in this case $(x, y, z, w) = (-9, 3, 6, 7)$. So the answer is $\boxed{7}$. ■

Appendix: In fact originally we intended to have the assumption that *all integers Monika has are positive.* The word "positive" was removed from the formulation of the problem accidentally at the very last minute. Here just for completeness, we give the solution of the problem if we also assume that all integers are positive (this solution was distributed at the day of the contest). In this case the solution is the same as above until the last paragraph starting with "In particular, the greatest integer ..". This paragraph should be removed and the rest of the solution has to proceed as follows:

Now since a, b, c, d are distinct,

$$a^2 + b^2 + c^2 \leq (d-1)^2 + (d-2)^2 + (d-3)^2 = 3d^2 - 12d + 14$$

So we need

$$d^2 - 12d + 14 > 0$$

And since the roots of this quadratic are $6 \pm \sqrt{22}$, we need $d \geq 11$. Then choosing $c = 10, b = 9, a = 8$ works, as it gives

$$K = \frac{11^2 + 10^2 + 9^2 + 8^2}{3} = 122$$

and so $x = 1, y = 22, z = 41, w = 58$ works. And we can quickly check that this is the smallest possible value of w, because

$$w = K - a^2 > d^2 - a^2 \geq d^2 - (d-3)^2 = 6d - 9 \geq 6(11) - 9 = 57$$

So the answer the assumption that all integers are positive is $\boxed{58}$. ■

Solution 20. Consider the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

It's easy to see (for example, by the ratio test) that this sum converges. Moreover, we may do an index shift to get

$$\begin{aligned}
\mathcal{S} &= \sum_{n=2}^{\infty} \frac{(n-1)!}{(2n-2)!} \\
&= \sum_{n=2}^{\infty} \frac{n!}{n(2n-2)!} \\
&= \sum_{n=2}^{\infty} \frac{2(2n-1)n!}{(2n)!} \\
&= 4\sum_{n=2}^{\infty} \frac{n \cdot n!}{(2n)!} - 2\sum_{n=2}^{\infty} \frac{n!}{(2n)!} \\
&= 4\sum_{n=1}^{\infty} \frac{n \cdot n!}{(2n)!} - 2\sum_{n=1}^{\infty} \frac{n!}{(2n)!} - 1 \\
&= -2\mathcal{S} + 4\sum_{n=1}^{\infty} \frac{n \cdot n!}{(2n)!} - 1
\end{aligned}$$

Thus, rearranging gives

$$4\mathcal{T} - 3\mathcal{S} = 1$$

where

$$\mathcal{T} = \sum_{n=1}^{\infty} \frac{n \cdot n!}{(2n)!}$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{n! \left(n - \frac{3}{4}\right)}{(2n)!} = \mathcal{T} - \frac{3}{4}\mathcal{S} = \boxed{\frac{1}{4}}$$

■