EF Exam Solutions
Texas A&M High School Math Contest
November 12, 2022

All answers must be simplified, and if units are involved, be sure to include them.

1. Find the maximum value of the function $f(x) = (\sin^2 x + 3)(2\cos^2 x + 3)$.

   Solution: If we denote $\cos^2 x = y$ we obtain

   $$f(x) = g(y) = (4 - y)(2y + 3) = -2y^2 + 5y + 12.$$

   Since $y \in [0, 1]$ and $g(y)$ is increasing on $(-\infty, \frac{5}{4}]$ (therefore on $[0, 1]$), we get that the maximum value of $f(x)$ is $g(1) = 15$.

   Answer: 15

2. If $\ln x + \ln y = \dfrac{13}{6}$ and $(\ln x)(\ln y) = 1$, find the value of $\log_x y + \log_y x$.

   Solution: We have that

   $$\log_x y + \log_y x = \frac{\ln y}{\ln x} + \frac{\ln x}{\ln y} = \frac{(\ln y)^2 + (\ln x)^2}{(\ln x)(\ln y)} = (\ln y)^2 + (\ln x)^2$$
   $$= (\ln x + \ln y)^2 - 2(\ln x)(\ln y) = \left(\frac{13}{6}\right)^2 - 2 = \frac{97}{36}.$$

   Answer: $\dfrac{97}{36}$

3. The curve $y = e^{2x} + 3$ intersects the y-axis at the point A, and the normal line to the curve at A intersects the x-axis at the point B. Find the distance from the origin O to the line AB.

   Solution: Let $f(x) = e^{2x} + 3$. Since $f(0) = 4$ and $f'(x) = 2e^{2x}$, we get that the coordinates of A are $(0, 4)$ and that the slope of the tangent line to the curve at A is $f'(0) = 2$. So the slope of the normal line to the curve at A is $-\frac{1}{2}$ and an equation for this line is $y - 4 = -\frac{1}{2}x$. When $y = 0$ we obtain $x = 8$ which is the x-coordinate of B. Then $OA = 4$, $OB = 8$ and $AB = 4\sqrt{5}$. Let M be the projection of O onto AB. Since $\triangle OAB$ is a right triangle we get that $OM \cdot AB = OA \cdot OB$ which implies that $OM = \dfrac{8}{\sqrt{5}} = \dfrac{8\sqrt{5}}{5}$.

   Answer: $\dfrac{8}{\sqrt{5}}$ or $\dfrac{8\sqrt{5}}{5}$

4. Find the exact value of

   $$\cos 1^\circ \cos 2^\circ \cos 3^\circ \cdots \cos 44^\circ \csc 45^\circ \csc 46^\circ \cdots \csc 89^\circ.$$

   Solution: Since

   $$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{\cos(90^\circ - \alpha)}$$

   we have that

   $$\csc 46^\circ = \frac{1}{\cos 44^\circ}, \quad \csc 47^\circ = \frac{1}{\cos 43^\circ}, \quad \cdots, \csc 89^\circ = \frac{1}{\cos 1^\circ}.$$

   Therefore our product is equal to $\csc 45^\circ = \sqrt{2}$.

   Answer: $\sqrt{2}$

5. A circle passes through the points $A(-4,0)$ and $B(0,-8)$. The center of the circle lies on the y-axis. Find the radius of the circle.

Solution: An equation of the circle has the form $(x-h)^2+(y-k)^2=r^2$, where the center has coordinates (h,k) and the radius is r. Since the center is on the y-axis we get that $h=0$. Taking into account that the points A and B are on the circle we obtain that $(-4)^2+(-k)^2=r^2$ and $(-8-k)^2=r^2$. These equations are equivalent to $16+k^2=r^2$ and $64+16k+k^2=r^2$. By subtracting the two equations we have that $16k+48=0 \Leftrightarrow k=-3$. This implies that $r^2=25 \Rightarrow r=5$.

Answer: 5

6. Find $\displaystyle\int_{-3}^{3}\left(\frac{\sin x}{\ln(5+x^2)}+\frac{1}{x+5}\right)dx$.

Solution: We notice that the function $f(x)=\dfrac{\sin x}{\ln(5+x^2)}$ is an odd function. That is, $f(-x)=-f(x)$, for all x. If we denote the integral $\displaystyle\int_{-3}^{3}\frac{\sin x}{\ln(5+x^2)}dx$ with I and we make the substitution $u=-x$, we get that $I=-I \Leftrightarrow I=0$. Therefore, our initial integral is equal to

$$\int_{-3}^{3}\frac{1}{x+5}dx=\ln(x+5)\Big|_{-3}^{3}=\ln 8-\ln 2=\ln 4=2\ln 2.$$

Answer: $\ln 4$ or $2\ln 2$

7. Determine m such that the polynomial $P(x)=2x^{29}+x^{23}+x^{12}+mx^{11}+x^8+5x^6+x^2+2$ is divisible by the polynomial $x^4+x^3+x^2+x+1$.

Solution: Let c be a root of $x^4+x^3+x^2+x+1$. Then $c^4+c^3+c^2+c+1=0$ which implies that

$$(c-1)(c^4+c^3+c^2+c+1)=c^5-1=0\Rightarrow c^5=1.$$

If $P(x)$ is divisible by $x^4+x^3+x^2+x+1$, then $P(c)=0$. We have that

$$\begin{aligned}P(c)&=2c^{29}+c^{23}+c^{12}+mc^{11}+c^8+5c^6+c^2+2=2c^4+c^3+c^2+mc+c^3+5c+c^2+2\\&=2c^4+2c^3+2c^2+(m+5)c+2=2(c^4+c^3+c^2+c+1)+(m+3)c=(m+3)c.\end{aligned}$$

Therefore, $(m+3)c=0$ which implies that $m=-3$ since $c\neq 0$.

Answer: $m=-3$

8. Find the sum

$$\ln\left(1+\frac{1}{2}\right)+\ln\left(1+\frac{1}{3}\right)+\ln\left(1+\frac{1}{4}\right)+\cdots+\ln\left(1+\frac{1}{2022}\right).$$

Solution: We have that

$$\ln\left(1+\frac{1}{n}\right)=\ln\left(\frac{n+1}{n}\right)=\ln(n+1)-\ln n,$$

for any positive integer n. So we get that

$$\begin{aligned}&\ln\left(1+\frac{1}{2}\right)+\ln\left(1+\frac{1}{3}\right)+\ln\left(1+\frac{1}{4}\right)+\cdots+\ln\left(1+\frac{1}{2022}\right)=\\&(\ln 3-\ln 2)+(\ln 4-\ln 3)+(\ln 5-\ln 4)+\cdots+(\ln(2023)-\ln(2022))=\\&\ln(2023)-\ln 2=\ln\left(\frac{2023}{2}\right).\end{aligned}$$

Answer: $\ln\left(\frac{2023}{2}\right)$

9. Let $f(x)$ be a one-to-one function such that $f(1) = 4$, $f(3) = 1$, $f'(1) = -4$, and $f'(3) = 2$. If $g(x) = f^{-1}(x)$ is the inverse function of $f(x)$, find the slope of the tangent line to the graph of $\frac{1}{g(x)}$ at $x = 1$.

Solution: The derivative of $\frac{1}{g(x)}$ is $-\frac{g'(x)}{(g(x))^2}$, so the slope of the tangent line is $-\frac{g'(1)}{(g(1))^2}$. Since $f(3) = 1$, we get that $g(1) = 3$ and $g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(3)} = \frac{1}{2}$. Therefore, the slope of the tangent line is $-\frac{1}{18}$.

Answer: $-\frac{1}{18}$

10. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2}$.

Solution: We notice that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{1-\left(\frac{k}{n}\right)^2} = \int_0^1 \sqrt{1-x^2}dx.$$

The graph of $y = \sqrt{1-x^2}, x \in [0,1]$ is the quarter from the first quadrant of the circle with center at the origin and radius 1. So, $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$.

Answer: $\frac{\pi}{4}$

11. Find x such that $\left(\sqrt[7]{5\sqrt{2}+7}\right)^x - \left(\sqrt[7]{5\sqrt{2}-7}\right)^x = 140\sqrt{2}$.

Solution: Let $\left(\sqrt[7]{5\sqrt{2}+7}\right)^x = y$. Since $(5\sqrt{2}+7)(5\sqrt{2}-7) = 1$, our equation becomes $y - \frac{1}{y} - 140\sqrt{2} = 0$ which is equivalent to $y^2 - 140\sqrt{2}y - 1 = 0$. This quadratic equation has solutions $y = 70\sqrt{2} \pm 99$ and from the fact that $y > 0$ we get that $y = 70\sqrt{2} + 99 = (5\sqrt{2}+7)^2$. Therefore, $\frac{x}{7} = 2 \Leftrightarrow x = 14$.

Answer: 14

12. Find the 2022th derivative of $f(x) = \sin^4 x + \cos^4 x$.

Solution: We see that

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x \cos x - 4\cos^3 x \sin x = 4 \sin x \cos x(\sin^2 x - \cos^2 x)\\ &= -2\sin(2x)\cos(2x) = -\sin(4x).\end{aligned}$$

Since the 2021th derivative of $\sin x$ is $\cos x$, we get that the 2022th derivative of $f(x)$ is $-4^{2021}\cos(4x)$.

Answer: $-4^{2021}\cos(4x)$

13. Let $f(x)$ be a differentiable function such that

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt,$$

for all real numbers x. Find $f(6)$.

Solution: We notice that $f(0) = 0$. If we make the substitution $u = x - t$ we get that

$$\begin{aligned} f(x) &= x^2 - \int_x^0 e^{u-x} f(u)du \\ &= x^2 + e^{-x} \int_0^x e^u f(u)du \Leftrightarrow \\ e^x f(x) &= x^2 e^x + \int_0^x e^u f(u)du. \end{aligned}$$

If we differentiate both sides we obtain

$$e^x f(x) + e^x f'(x) = 2xe^x + x^2 e^x + e^x f(x)$$

which is equivalent to $f'(x) = x^2 + 2x$. This implies that $f(x) = \frac{1}{3}x^3 + x^2 + c$, and since $f(0) = 0$ we get that $c = 0$. Therefore, $f(x) = \frac{1}{3}x^3 + x^2$ and $f(6) = 108$.

Answer: 108

14. Consider the expression

$$E(n) = \frac{1}{2n+1}\binom{2n}{0} - \frac{1}{2n}\binom{2n}{1} + \frac{1}{2n-1}\binom{2n}{2} + \cdots - \frac{1}{2}\binom{2n}{2n-1} + \binom{2n}{2n},$$

where n is a positive integer. Find $E(2022)$.

Solution: We know that

$$(x-1)^{2n} = \binom{2n}{0}x^{2n} - \binom{2n}{1}x^{2n-1} + \cdots - \binom{2n}{2n-1}x + \binom{2n}{2n}.$$

If we integrate both sides on $[0, 1]$ we get

$$\left.\frac{(x-1)^{2n+1}}{2n+1}\right|_0^1 = \binom{2n}{0}\left.\frac{x^{2n+1}}{2n+1}\right|_0^1 - \binom{2n}{1}\left.\frac{x^{2n}}{2n}\right|_0^1 + \cdots - \binom{2n}{2n-1}\left.\frac{x^2}{2}\right|_0^1 + \binom{2n}{2n}\left.x\right|_0^1,$$

which implies that $E(n) = \dfrac{1}{2n+1}$. Therefore, $E(2022) = \dfrac{1}{4045}$

Answer: $\dfrac{1}{4045}$

15. Find the sum of all solutions $\theta \in \left[0, \dfrac{\pi}{2}\right]$ of the equation

$$\frac{\sin 2\theta + \sin 4\theta + \sin 6\theta + \sin 8\theta}{\cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta} = 1.$$

Solution: We obtain first the following identity

$$\frac{(\sin 2\theta + \sin 8\theta) + (\sin 4\theta + \sin 6\theta)}{(\cos 2\theta + \cos 8\theta) + (\cos 4\theta + \cos 6\theta)} = \frac{2\sin 5\theta \cos 3\theta + 2\sin 5\theta \cos\theta}{2\cos 5\theta \cos 3\theta + 2\cos 5\theta \cos\theta} = \frac{2\sin 5\theta(\cos 3\theta + \cos\theta)}{2\cos 5\theta(\cos 3\theta + \cos\theta)} = \tan 5\theta.$$

So our equation becomes $\tan 5\theta = 1$ and since $5\theta \in \left[0, \dfrac{5\pi}{2}\right]$ we get that 5θ is equal to $\dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$ or $\dfrac{9\pi}{4}$. Therefore, the solutions of the equation in $\left[0, \dfrac{\pi}{2}\right]$ are $\dfrac{\pi}{20}, \dfrac{5\pi}{20}$, and $\dfrac{9\pi}{20}$ and their sum is $\dfrac{15\pi}{20} = \dfrac{3\pi}{4}$.

Answer: $\dfrac{3\pi}{4}$

16. Find $\lim\limits_{x\to 0} \dfrac{\sin(x^{2022}) - (\sin x)^{2022}}{x^{2024}}$.

Solution: We use the Taylor expansion $\sin x = x - \dfrac{x^3}{6} + o(x^3)$.

$$\sin(x^{2022}) = x^{2022} - \frac{x^{6066}}{6} + o\left(x^{6066}\right).$$

and using also the binomial formula:

$$(\sin x)^{2022} = \left(x - \frac{x^3}{6} + o(x^3)\right)^{2022} = x^{2022} - \frac{2022}{6}x^{2021}x^3 + O(x^{2020}x^6) = x^{2022} - 337x^{2024} + o(x^{2024}).$$

Therefore $\dfrac{\sin(x^{2022}) - (\sin x)^{2022}}{x^{2024}} = \dfrac{337x^{2024} + o(x^{2024})}{x^{2024}} \xrightarrow[x\to 0]{} 337.$

Answer: 337

17. Consider the sequence $\{a_n\}$ where $a_n = \displaystyle\sum_{k=1}^{n} \frac{k^2+k}{n^3+k}$ for every positive integer n. Find $\lim\limits_{n\to\infty} a_n$.

Solution: Let $b_n = \displaystyle\sum_{k=1}^{n} \frac{k^2+k}{n^3}$ and $c_n = \displaystyle\sum_{k=1}^{n} \frac{k^2+k}{n^3+n}$. We see that $c_n \le a_n \le b_n$ for every positive integer n. Next we find the limits of $\{b_n\}$ and $\{c_n\}$. We have that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}(k^2+k) = \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{1}{3}$$

and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n^3+n}\sum_{k=1}^{n}(k^2+k) = \lim_{n\to\infty} \frac{1}{n^3+n}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{1}{3}.$$

By the Squeeze Theorem, we get that $\lim\limits_{n\to\infty} a_n = \dfrac{1}{3}$.

Answer: $\dfrac{1}{3}$

18. Let $a_n = \displaystyle\int_{\frac{1}{n+1}}^{\frac{1}{n}} \arctan(nx)dx$ and $b_n = \displaystyle\int_{\frac{1}{n+1}}^{\frac{1}{n}} \arcsin(nx)dx$. Find $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$.

Solution: From the Mean Value Theorem for integrals we know that there exists x_n between $\dfrac{1}{n+1}$ and $\dfrac{1}{n}$ such that

$$a_n = \arctan(nx_n)\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

The function $\arctan x$ is increasing, so we have that

$$\arctan\left(\frac{n}{n+1}\right)\frac{1}{n(n+1)} < a_n < \arctan\left(\frac{n}{n}\right)\frac{1}{n(n+1)} \Leftrightarrow$$
$$\arctan\left(\frac{n}{n+1}\right) < n(n+1)a_n < \arctan 1.$$

By the Squeeze Theorem, we obtain that $\lim_{n\to\infty} n(n+1)a_n = \arctan 1 = \frac{\pi}{4}$. Similarly, $\lim_{n\to\infty} n(n+1)b_n = \arcsin 1 = \frac{\pi}{2}$. Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$.

Answer: $\frac{1}{2}$

19. Evaluate the integral

$$I = \int_0^1 \frac{\sin^2\left(\frac{\pi x^2}{2}\right)}{\sqrt{1-x^2}}dx.$$

Solution: First we substitute $x = \sin\theta$ to get that

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}\sin^2\theta\right) d\theta.$$

Then we make the substitution $\alpha = \frac{\pi}{2} - \theta$ to write the integral as

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}\cos^2\alpha\right) d\alpha = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}\cos^2\theta\right) d\theta.$$

From here we notice that

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}\cos^2\theta\right) d\theta = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - \frac{\pi}{2}\sin^2\theta\right) d\theta = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2}\sin^2\theta\right) d\theta.$$

Therefore, we have

$$2I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}\sin^2\theta\right) d\theta + \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2}\sin^2\theta\right) d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2},$$

which implies that $I = \frac{\pi}{4}$.

Answer: $\frac{\pi}{4}$

20. A dart is thrown at (and hits) a square dartboard. Assuming each spot on the dartboard has an equal chance of being hit, find the probability that the dart lands at a point closer to the center of the board than any of the edges. Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are integers.

Solution: Position the square such that the vertices are at $(\pm 1, \pm 1)$. Two diagonals divide the square into 4 triangles. By symmetry consider the triangle whose vertices are $(-1, 1)$, $(0, 0)$, and $(1, 1)$. The dart must hit a point (x, y) such that $\sqrt{x^2+y^2} \le 1-y$ which is equivalent to $y \le \frac{1}{2}(1-x^2)$. Since the triangle has area 1, the probability is equal to the area between the graphs of $y = \frac{1}{2}(1-x^2)$ and $y = |x|$, which intersect at points with x-coordinates $\pm(\sqrt{2}-1)$. Both functions are even, so the area is

$$2\int_0^{\sqrt{2}-1} \frac{1}{2}(1-x^2-2x)dx = \left(x - \frac{x^3}{3} - x^2\right)\Big|_0^{\sqrt{2}-1}$$

$$(\sqrt{2}-1) - \frac{1}{3}(2\sqrt{2}-6+3\sqrt{2}-1) - (2-2\sqrt{2}+1) = \frac{-5+4\sqrt{2}}{3}.$$

Answer: $\frac{-5+4\sqrt{2}}{3}$