EF Exam
Texas A&M High School Math Contest
November 12, 2022

Directions: *All answers should be simplified.*

1. Find the maximum value of the function $f(x) = (\sin^2 x + 3)(2\cos^2 x + 3)$.

2. If $\ln x + \ln y = \dfrac{13}{6}$ and $(\ln x)(\ln y) = 1$, find the value of $\log_x y + \log_y x$.

3. The curve $y = e^{2x} + 3$ intersects the y-axis at the point A, and the normal line to the curve at A intersects the x-axis at the point B. Find the distance from the origin O to the line AB.

4. Find the exact value of
$$\cos 1^\circ \cos 2^\circ \cos 3^\circ \cdots \cos 44^\circ \csc 45^\circ \csc 46^\circ \cdots \csc 89^\circ.$$

5. A circle passes through the points $A(-4, 0)$ and $B(0, -8)$. The center of the circle lies on the y-axis. Find the radius of the circle.

6. Find $\displaystyle\int_{-3}^{3} \left(\frac{\sin x}{\ln(5 + x^2)} + \frac{1}{x + 5} \right) dx$.

7. Determine m such that the polynomial $P(x) = 2x^{29} + x^{23} + x^{12} + mx^{11} + x^8 + 5x^6 + x^2 + 2$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

8. Find the sum
$$\ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) + \ln\left(1 + \frac{1}{4}\right) + \cdots + \ln\left(1 + \frac{1}{2022}\right).$$

9. Let $f(x)$ be a one-to-one function such that $f(1) = 4$, $f(3) = 1$, $f'(1) = -4$, and $f'(3) = 2$. If $g(x) = f^{-1}(x)$ is the inverse function of $f(x)$, find the slope of the tangent line to the graph of $\dfrac{1}{g(x)}$ at $x = 1$.

10. Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2 - k^2}}{n^2}$.

11. Find x such that $\left(\sqrt[7]{5\sqrt{2} + 7}\right)^x - \left(\sqrt[7]{5\sqrt{2} - 7}\right)^x = 140\sqrt{2}$.

12. Find the 2022th derivative of $f(x) = \sin^4 x + \cos^4 x$.

13. Let $f(x)$ be a differentiable function such that
$$f(x) = x^2 + \int_0^x e^{-t} f(x - t)dt,$$
for all real numbers x. Find $f(6)$.

14. Consider the expression
$$E(n) = \frac{1}{2n+1}\binom{2n}{0} - \frac{1}{2n}\binom{2n}{1} + \frac{1}{2n-1}\binom{2n}{2} + \cdots - \frac{1}{2}\binom{2n}{2n-1} + \binom{2n}{2n},$$
where n is a positive integer. Find $E(2022)$.

15. Find the sum of all solutions $\theta \in \left[0, \frac{\pi}{2}\right]$ of the equation

$$\frac{\sin 2\theta + \sin 4\theta + \sin 6\theta + \sin 8\theta}{\cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta} = 1.$$

16. Find $\lim_{x \to 0} \frac{\sin(x^{2022}) - (\sin x)^{2022}}{x^{2024}}$.

17. Consider the sequence $\{a_n\}$ where $a_n = \sum_{k=1}^{n} \frac{k^2 + k}{n^3 + k}$ for every positive integer n. Find $\lim_{n \to \infty} a_n$.

18. Let $a_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \arctan(nx)dx$ and $b_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \arcsin(nx)dx$. Find $\lim_{n \to \infty} \frac{a_n}{b_n}$.

19. Evaluate the integral

$$I = \int_0^1 \frac{\sin^2\left(\frac{\pi x^2}{2}\right)}{\sqrt{1 - x^2}} dx.$$

20. A dart is thrown at (and hits) a square dartboard. Assuming each spot on the dartboard has an equal chance of being hit, find the probability that the dart lands at a point closer to the center of the board than any of the edges. Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where a, b, c and d are integers.