MATH 151, FALL 2021
COMMON EXAM II - VERSION B

LAST NAME(print): ______________________ FIRST NAME(print): ______________________

INSTRUCTOR: ______________________

SECTION NUMBER: ____________

DIRECTIONS:

1. No calculator, cell phones, or other electronic devices may be used, and they must all be put away out of sight.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam on the ScanTron form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ______________________________________

PART I: Multiple Choice. 3 points each

Suppose f and g are differentiable functions which satisfy the following condition. Use the following table for Questions 1 and 2.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	-1	3	5	0
1	1	2	-1	2

1. Find $H'(1)$ for $H(x) = \dfrac{f(g(x))}{x^2}$.

   (a) 0
   (b) 2
   (c) 4
   (d) 6
   (e) 8 $\leftarrow$ correct

2. Find $h'(1)$ for $h(x) = x^2 f(g(x))$.

   (a) 1
   (b) 4 $\leftarrow$ correct
   (c) 6
   (d) 8
   (e) 12

3. Which of the statements is true about $f(x)$?

$$f(x) = \begin{cases} e^x + 4, & x < 0 \\ x^3 + x + 5, & 0 \le x \le 2 \\ 10x - 5, & 2 < x \end{cases}$$

   (a) f is continuous but not differentiable at both $x = 0$ and $x = 2$.
   (b) f is continuous but not differentiable at $x = 0$; f is differentiable at $x = 2$.
   (c) f is differentiable at both $x = 0$ and $x = 2$.
   (d) f is differentiable at $x = 0$; f is continuous but not differentiable at $x = 2$. $\leftarrow$ correct
   (e) f is not continuous at $x = 0$ or $x = 2$.

4. Find $f''(x)$ for $f(x) = e^{1/x}$.

   (a) $\left(\frac{2}{x^3} - \frac{1}{x^2}\right)e^{1/x}$

   (b) $\left(\frac{2}{x^3} - \frac{1}{x^3}\right)e^{1/x}$

   (c) $\left(\frac{1}{x^2}\right)e^{1/x}$

   (d) $\left(-\frac{2}{x^5}\right)e^{1/x}$

   (e) $\left(\frac{2}{x^3} + \frac{1}{x^4}\right)e^{1/x}$ $\leftarrow$ correct

5. Find the derivative of the function $f(x) = \arcsin\left(3^{6x}\right)$

   (a) $\frac{6 \cdot 3^{6x}}{\sqrt{1 - 3^{12x}}}$

   (b) $-\frac{6 \cdot 3^{6x}}{\sqrt{1 - 3^{12x}}}$

   (c) $\frac{6 \ln 3 \cdot 3^{6x}}{1 + 3^{12x}}$

   (d) $-\frac{6 \ln 3 \cdot 3^{6x}}{\sqrt{1 - 3^{12x}}}$

   (e) $\frac{6 \ln 3 \cdot 3^{6x}}{\sqrt{1 - 3^{12x}}}$ $\leftarrow$correct

6. Find the 2021^{th} derivative of $f(x) = 2xe^{-x}$.

   (a) $e^{-x}(-x + 2021)$

   (b) $e^{-x}(x - 2021)$

   (c) $2e^{-x}(x - 2021)$

   (d) $2e^{-x}(-x + 2021)$ $\leftarrow$ correct

   (e) $2^{2021}e^{-x}(-x + 2021)$

7. Find all the value(s) of x on the interval $[0, 2\pi]$ for which the tangent line to the graph of $f(x) = \sin^2 x - \cos x$ is horizontal.

   (a) $0, \pi, 2\pi$
   (b) $0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}$
   (c) $0, \pi, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ $\leftarrow$ correct
   (d) $0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$
   (e) $0, \pi, 2\pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

8. A ball is thrown vertically upward with a velocity of 40 feet per second and the height, s, of the ball at time t seconds is given by $s(t) = 40t - 8t^2$. What is the velocity of the ball when it is 48 feet above the ground on its way up?

   (a) 12 ft/sec
   (b) 24 ft/sec
   (c) 32 ft/sec
   (d) 54 ft/sec
   (e) 8 ft/sec $\leftarrow$ correct

9. Let $g(x) = f(x^2 + 1)$. Given the table of values below for f and f', find the equation of the tangent line to $g(x)$ at $x = 1$.

x	$f(x)$	$f'(x)$
1	3	4
2	5	6
3	1	2

   (a) $y - 5 = -2(x - 1)$
   (b) $y - 3 = 4(x - 1)$
   (c) $y - 3 = 6(x - 1)$
   (d) $y - 5 = 6(x - 1)$
   (e) $y - 5 = 12(x - 1)$ $\leftarrow$ correct

10. Find the slope of the tangent line to the curve $y^3 - xy = 2x + 4$ at the point $(1, 2)$.

(a) $\dfrac{1}{3}$

(b) 8

(c) $\dfrac{4}{11}$ $\leftarrow$ correct

(d) $\dfrac{22}{11}$

(e) $\dfrac{5}{3}$

11. Find $f'(x)$ for $f(x) = \ln\left(\dfrac{\sqrt{x^4 - 3}}{\sec^2 x}\right)$

(a) $\dfrac{1}{8x^3} - \dfrac{2}{\sec x \tan x}$

(b) $\dfrac{1}{8x^3} - \dfrac{1}{2 \sec x \tan x}$

(c) $\dfrac{2x^3}{x^4 - 3} - 2 \tan x$ $\leftarrow$ correct

(d) $\dfrac{2x^3}{x^4 - 3} - \dfrac{2 \tan x}{\sec x}$

(e) $\dfrac{2x^3}{x^4 - 3} - \tan x$

12. Find the derivative of $f(x) = \log_{10}(x^5 + 5^x)$.

(a) $\dfrac{5x^4 + 5^x}{(x^5 + 5^x) \ln 10}$

(b) $\dfrac{5x^4}{(x^5 + 5^x) \ln 10}$

(c) $\dfrac{5x^4 + 5^x \ln 5}{x^5 + 5^x}$

(d) $\dfrac{5x^4 + 5^x \ln 5}{(x^5 + 5^x) \ln 10}$ $\leftarrow$ correct

(e) $\dfrac{5x^4 + 5^x}{x^5 + 5^x}$

13. Which of the following vectors is tangent to the curve $\mathbf{r}(t) = \left\langle \sqrt{t^2+1}, t \right\rangle$ when $t = \sqrt{3}$?

(a) $\left\langle \dfrac{\sqrt{3}}{4}, 1 \right\rangle$

(b) $\left\langle \dfrac{\sqrt{3}}{2}, 1 \right\rangle$ $\leftarrow$ correct

(c) $\left\langle \dfrac{1}{2}, 1 \right\rangle$

(d) $\left\langle \dfrac{1}{\sqrt{5}}, 1 \right\rangle$

(e) $\left\langle \dfrac{2}{\sqrt{5}}, 1 \right\rangle$

14. Let $\mathbf{r}(t) = \langle \sin(2t), \cos t \rangle$. For which of the following points does the curve have a vertical tangent line?

(a) $P = \left(1, \dfrac{\sqrt{2}}{2}\right)$ $\leftarrow$ correct

(b) $P = \left(1, \dfrac{\sqrt{3}}{2}\right)$

(c) $P = \left(\dfrac{\sqrt{3}}{2}, 1\right)$

(d) $P = \left(\dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right)$

(e) $P = (0, 1)$

15. Find the slope of the tangent line at the point $(2, 0)$ for the following parametric curve

$$x(t) = t^4 + 1, \quad y(t) = \cos\left(\frac{\pi t}{2}\right)$$

(a) $-\dfrac{8}{\pi}$

(b) $-\dfrac{1}{4}$

(c) $-\dfrac{\pi}{8}$ $\leftarrow$ correct

(d) 0

(e) $-\dfrac{\pi}{4}$

16. The position function of an object moving along a straight line is given by $s(t) = t^4 - 2t^2 + 1$, where position is measured in feet and time in seconds. Find the total distance traveled by the object during the first two seconds.

(a) 8

(b) 9

(c) 10 $\leftarrow$ correct

(d) 11

(e) 12

17. A sample of a radioactive substance decayed to $\frac{1}{5}$ of its original amount after 9 years. What is the half-life of the substance? The half-life is the amount of time needed to decay to half of its original amount.

(a) $\dfrac{\ln \frac{1}{5}}{9 \ln \frac{1}{2}}$ years

(b) $-\dfrac{9 \ln \frac{1}{5}}{\ln \frac{1}{2}}$ years

(c) $\dfrac{9 \ln \frac{1}{5}}{\ln \frac{1}{2}}$

(d) $\dfrac{9 \ln \frac{1}{2}}{\ln \frac{1}{5}}$ years $\leftarrow$ correct

(e) $-\dfrac{9 \ln \frac{1}{2}}{\ln \frac{1}{5}}$ years

18. A cubic block of ice (which remains in the shape of a cube) is melting so that its volume is decreasing at a rate of 4 cm^3/min. How fast is the length of a side changing (in cm/min) when the sides are 10 cm?

(a) $\dfrac{4}{300}$

(b) $\dfrac{1}{1200}$

(c) 0

(d) $-\dfrac{1}{1200}$

(e) $-\dfrac{4}{300}$ $\leftarrow$ correct

19. Use a linear approximation (or differentials) at $x = 8$ to approximate the value of $\sqrt[3]{8.1}$.

(a) $\dfrac{241}{120}$ $\leftarrow$ correct

(b) $\dfrac{121}{60}$

(c) $\dfrac{481}{240}$

(d) $\dfrac{61}{30}$

(e) $\dfrac{961}{480}$

20. The radius of a circle is measured to be 1 meter with a possible error of 0.04m. Use differentials or linear approximation to estimate the maximum possible error in the area of the circle.

(a) 0.08π $\leftarrow$ correct

(b) 0.06π

(c) 0.04π

(d) 0.10π

(e) 0.05π

PART II WORK OUT

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

21. (12 points) Find $\dfrac{dy}{dx}$ for the following functions. **Do not simplify after taking the derivative**.

(a) $y = \tan^4\big(\cos(3x)\big)$

Sol. Aplly Power Rule and Chain Rule to have

$$\begin{aligned} y' =& 4\tan^3(\cos 3x)\Big(\tan(\cos 3x)\Big)' \\ =& 4\tan^3(\cos 3x)\sec^2(\cos 3x)\Big(\cos 3x\Big)' \\ =& 4\tan^3(\cos 3x)\sec^2(\cos 3x)(-3\sin 3x) \end{aligned}$$

(b) $y = \dfrac{e^{2x}\tan x}{x^3+\pi^3}$

Sol. Aplly Qutient Rule and Chain Rule to have

$$\begin{aligned} y' =& \frac{\big(e^{2x}\tan x)'(x^3+\pi^3) - e^{2x}\tan x\big(x^3+\pi^3\big)'}{(x^3+\pi^3)^2} \\ =& \frac{\big(2e^{2x}\tan x + e^{2x}\sec^2 x)(x^3+\pi^3) - e^{2x}\tan x\big(3x^2\big)}{(x^3+\pi^3)^2} \end{aligned}$$

Alternatively, one can apply Logarithmic Differentiation.

$$\begin{aligned} \ln y = \ln\left(\frac{e^{2x}\tan x}{x^3+\pi^3}\right) &= 2x + \ln\tan x - \ln(x^3+\pi^3) \\ \Rightarrow \quad \frac{y'}{y} =& 2 + \frac{\sec^2 x}{\tan x} - \frac{3x^2}{x^3+\pi^3} \\ \Rightarrow \quad y' =& \left(2 + \frac{\sec^2 x}{\tan x} - \frac{3x^2}{x^3+\pi^3}\right)\frac{e^{2x}\tan x}{x^3+\pi^3} \end{aligned}$$

22. (7 points) Find $\dfrac{dy}{dx}$. Your answer must be a function in x only.

$y = \big(\sin(3x)\big)^{2\sqrt{x}}$

Sol. Take ln of $y = \big(\sin 3x\big)^{2\sqrt{x}}$ to have

$$\ln y = \ln\big(\sin 3x\big)^{2\sqrt{x}} = 2\sqrt{x}\cdot\ln(\sin 3x)$$

Apply Impicit Differentiation to have

$$\begin{aligned}\frac{y'}{y} =& \big(2\sqrt{x}\big)'\cdot\ln(\sin 3x) + 2\sqrt{x}\cdot\big(\ln(\sin 3x)\big)' \\ =& (2\cdot\frac{1}{2}x^{-1/2})\ln(\sin 3x) + 2\sqrt{x}\cdot\frac{1}{\sin 3x}(\cos 3x)3\end{aligned}$$

Thus we have

$$y' = \Big(\frac{1}{\sqrt{x}}\ln(\sin 3x) + 2\sqrt{x}\cdot\frac{1}{\sin 3x}(\cos 3x)3\Big)\big(\sin 3x\big)^{2\sqrt{x}}$$

Acceptible answers include

$$\begin{aligned}y =& \Big(\frac{1}{\sqrt{x}}\ln(\sin 3x) + 2\cdot 3\sqrt{x}\cdot\frac{\cos 3x}{\sin 3x}\Big)\big(\sin 3x\big)^{2\sqrt{x}} \\ =& \Big(\frac{1}{\sqrt{x}}\ln(\sin 3x) - 6\sqrt{x}\cot 3x\Big)\big(\sin 3x\big)^{2\sqrt{x}}\end{aligned}$$

23. (8 points) There are two lines tangent to the parabola $y = 2x^2$ that pass through the point $(2, -10)$.

(a) Find the x-coordinates where these tangent lines touch the parabola.

Sol. Let $(a, 2a^2)$ be a tangent point on the parabola. Since $f'(x) = 4x$, we have an equation of the tangent line

$$y - 2a^2 = 4a(x - a) \tag{1}$$

The tangent line also passes $(2, -10)$. Thus we have

$$-10 - 2a^2 = 4a(2 - a) \quad \Rightarrow \quad 2a^2 - 8a - 10 = 0$$

Solving the equation we have

$$a^2 - 4a - 5 = (a + 1)(a - 5) = 0 \quad \Rightarrow \quad a = -1, 5$$

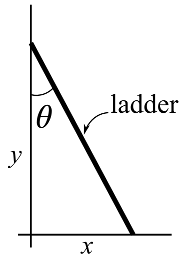
So the x-coordinates are -1 and 5.

(b) Find equations of the two tangent lines.

Sol. By (a) above we see that tangent points are $(-1, 2)$ and $(5, 50)$. The equation of the tangent line (1) above becomes

$$y - 2 = -4(x + 1) \text{ and } y - 50 = 20(x - 5)$$

24. (13 points) A ladder 10 feet long is leaning against a vertical wall. The base of the ladder is pulled away from the wall at a rate of 3 ft/s.



(a) How fast is the top of the ladder sliding down the wall when the base is 8 feet from the wall?

Sol. Let $x(t)$ and $y(t)$ denote the distance as in the figure above. By taking the derivative of equation $x^2 + y^2 = 10^2$ with respect to t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \text{ or } x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

The condition gives $\frac{dx}{dt} = 3$. Moreover, at the moment the base is 8 feet from the wall, we have $x = 8$ and $y = \sqrt{10^2 - 8^2} = 6$. Plugging those values in the above equation implies

$$8 \cdot 3 + 6\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{24}{6} = -4$$

Thus the top of the ladder slides down the wall at a rate 4 ft/s at the moment.

(b) Find the rate at which the angle θ betweeen the ladder and the wall is changing when the base of the ladder is 8 feet from the wall.

Sol. Take the derivative of the following equation with respect to t.

$$\sin\theta = \frac{x}{10}$$

The resulting eqaution is

$$\cos\theta\,\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}.$$

Since $\frac{dx}{dt} = 3$, and $\cos\theta = \frac{6}{10}$, we have

$$\frac{6}{10}\frac{d\theta}{dt} = \frac{1}{10}3 \quad \Rightarrow \frac{d\theta}{dt} = \frac{3}{6} = \frac{1}{2}\text{rad/s}.$$

One can also use

$$\cos\theta = \frac{y}{10} \Rightarrow -\sin\theta\,\frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}.$$

Plugging in $\frac{dy}{dt} = -4$, and $\sin\theta = \frac{8}{10}$, we have $\frac{d\theta}{dt} = \frac{1}{2}\text{rad/s}.$