MATH 151, FALL 2021

COMMON EXAM III - VERSION B

LAST NAME(print): ______________________________FIRST NAME(print): ______________________________

INSTRUCTOR: ______________________________

SECTION NUMBER: _______________

DIRECTIONS:

1. No calculator, cell phones, or other electronic devices may be used, and they must all be put away out of sight.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam on the ScanTron form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ____

PART I: Multiple Choice. 4 points each

1. Find the absolute maximum M and absolute minimum m of $\dfrac{x}{x^2+9}$ on $[0,6]$.

   (a) $M = \dfrac{2}{15}$, $m = 0$
   (b) $M = \dfrac{1}{6}$, $m = -\dfrac{1}{6}$
   (c) $M = \dfrac{1}{6}$, $m = \dfrac{2}{15}$
   (d) $M = \dfrac{1}{6}$, $m = 0$ $\leftarrow$ correct
   (e) None of the above

2. Consider the following piecewise function f on $[-2,2]$. Find the absolute maximum and absolute minimum of f.

$$f(x) = \begin{cases} x^2 - 4 & -2 \le x < 0 \\ -x + 1 & 0 \le x \le 2 \end{cases}$$

   (a) Absolute maximum is 1 and absolute minimum is -4.
   (b) Absolute maximum is 1 and absolute minimum is -1.
   (c) f doesn't have either since f is not continuous on the interval.
   (d) Absolute maximum is 1 and f doesn't admit absolute minimum. $\leftarrow$ correct
   (e) None of the above

3. Find the critical numbers of the function.

$$F(x) = x^{\frac{2}{3}}(x-4)^2$$

   (a) $x = 0, 1, 4$ $\leftarrow$ correct
   (b) $x = 1, 4$
   (c) $x = 0, \dfrac{8}{5}$
   (d) $x = \dfrac{8}{5}$
   (e) $x = 0, \dfrac{4}{3}$

4. Find the t values of all local extrema of $f(t) = 2\cos t + \sin 2t$ on $[0, \pi]$.
Hint: You can use the identity $\cos 2t = 1 - 2\sin^2 t$

(a) $t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

(b) $t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

(c) $t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ ← correct

(d) $t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

(e) $t = 0, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

5. Find the most general antiderivative of $f(x) = \dfrac{-2}{\sqrt{1-x^2}} + 3^x + \sqrt[3]{x^2}$.

(a) $F(x) = 2\arcsin x + \dfrac{3^x}{\ln 3} + \dfrac{3}{5}x^{5/3} + C$

(b) $F(x) = 2\arcsin x + \dfrac{3^x}{\ln 3} + \dfrac{2}{5}x^{5/2} + C$

(c) $F(x) = -2\arcsin x + 3^x + \dfrac{3}{5}x^{5/3} + C$

(d) $F(x) = -2\arcsin x + \dfrac{3^x}{\ln 3} + \dfrac{3}{5}x^{5/3} + C$ ← correct

(e) None of the above

6. Find the INCORRECT statement for $f(x) = x\sqrt{4-x}$.

(a) The domain of f is $(-\infty, 4]$

(b) f is continuous on $[0, 4]$ but not differentiable on $(0, 4)$ ← correct

(c) f satisfies the hypotheses of the Mean Value Theorem on $[0, 4]$

(d) $f'\left(\dfrac{8}{3}\right) = \dfrac{f(4) - f(0)}{4 - 0}$

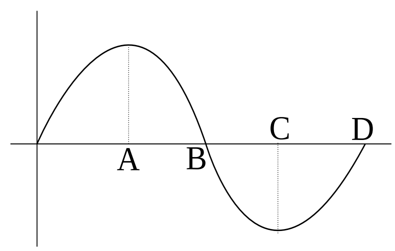
(e) f has a local maximum at $x = \dfrac{8}{3}$

7. Find where $g(x) = e^{-x^2}$ is concave downward.

   (a) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$
   (b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ ← correct
   (c) $(-\infty, 0)$
   (d) $(0, \infty)$
   (e) $\left(\frac{1}{\sqrt{2}}, \infty\right)$

8. If the graph below is the FIRST DERIVATIVE of a function f, on which of the following intervals is f both decreasing and concave up?

   (a) $(0, A)$
   (b) (A, B)
   (c) (B, C)
   (d) (B, D)
   (e) (C, D) ← correct



9. Suppose g is a continuous function defined on $(-\infty, \infty)$. Find the x-coordiante of inflection points of g if $g''(x) = -e^x(x-1)(x-3)^3(x-4)^4$.

   (a) $x = 0, 1, 3, 4$
   (b) $x = 0, 1, 3$
   (c) $x = 1, 3, 4$
   (d) $x = 1, 3$ ← correct
   (e) g does not have an inflection point

10. Find $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{e^x - 1} \right]$

(a) $\frac{1}{2}$ $\leftarrow$ correct

(b) 1

(c) 0

(d) -1

(e) $-\infty$

11. Consider the function f defined by $f(x) = \frac{x^2}{2} + 2\cos x$, for $0 \leq x \leq \pi$. Which of the following statements is true?

(a) The graph of f is concave upwards throughout the interval $(0, \pi)$

(b) The graph of f is concave upwards in the interval $\left(\frac{\pi}{6}, \frac{5\pi}{6} \right)$

(c) The graph of f is concave downwards in the interval $\left(0, \frac{\pi}{3} \right)$ $\leftarrow$ correct

(d) The graph of f is concave downwards throughout the interval $(0, \pi)$

(e) The graph of f is concave downwards in the interval $\left(0, \frac{\pi}{6} \right)$ and $\left(\frac{5\pi}{6}, \pi \right)$

12. The function $f(x)$ is defined at all real numbers except -3 and $f'(x) = \frac{-3(x-1)(x+1)}{(x+3)^3}$. At what x-value(s) does $f(x)$ have a local maximum?

(a) $x = -3$ only

(b) $x = -1$ only

(c) $x = 1$ only $\leftarrow$ correct

(d) $x = 1$ and $x = -3$

(e) $f(x)$ doesn't admit a local maximum.

13. Calculate $\lim_{x \to 0^+} \Big(1 + f(x)\Big)^{-2/x}$ if $f(x)$ has a continuous first derivative and satisfies $f(0) = 0$ and $f'(0) = 3$.

(a) -2

(b) -6

(c) e^{-3}

(d) e^{-4}

(e) e^{-6} $\leftarrow$ correct

14. Find INCORRECT statement for the behavior of the function $f(x) = 1 + \frac{3}{x} - \frac{2}{x^2}$, $x \neq 0$.

(a) f increases on $\left(0, \frac{4}{3}\right)$

(b) f decreases on $(-\infty, 0)$ and $\left(\frac{4}{3}, \infty\right)$

(c) $f\left(\frac{4}{3}\right)$ is a local maximum.

(d) f has no horizontal asymptote $\leftarrow$ correct

(e) f has the vertical asymptote $x = 0$

15. Which of the following gives the exact area under the curve $f(x) = \ln(x)$ on the interval $[2, 7]$?

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \ln\left(\frac{5}{n} i\right)$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7}{n} \ln\left(2 + \frac{5}{n} i\right)$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7}{n} \ln\left(\frac{7}{n} i\right)$

(d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7}{n} \ln\left(2 + \frac{7}{n} i\right)$

(e) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \ln\left(2 + \frac{5}{n} i\right)$ $\leftarrow$ correct

16. Approximate the area under the curve $f(x) = \sqrt{x+1}$ on $[0, 12]$ with 3 left endpoints.

(a) 12

(b) $16 + 4\sqrt{5}$ ← correct

(c) $12 + 4\sqrt{5} + 4\sqrt{13}$

(d) $4\sqrt{3} + 4\sqrt{7} + 4\sqrt{11}$

(e) $16 + 4\sqrt{5} + 4\sqrt{13}$

PART II WORK OUT

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

17. (6 points) Find the most general antiderivative of

$f'(x) = \frac{6}{x} + \sec x \cdot \tan x + 5\sqrt[4]{x^3} + \frac{2}{1+x^2}$

Sol. The most general antiderivative of f' is

$$\begin{aligned} F(x) =& 6\ln|x| + \sec x + 5\frac{4}{7}x^{7/4} + 2\arctan x + C \\ =& 6\ln|x| + \sec x + \frac{20}{7}x^{7/4} + 2\arctan x + C \end{aligned}$$

18. (8 points) The acceleration of object is given by $a(t) = -\dfrac{1}{t^2}$, $t > 0$. Given the positions $s(1) = 2$ and $s(2) = 2$, find the position function $s(t)$.

Sol. Taking an antiderivative of $a(t) = -t^{-2}$ yields

$$s'(t) = v(t) = \frac{1}{t} + C_1 \quad \Rightarrow \quad s(t) = \ln|t| + C_1 t + C_2$$

for some constants C_1 and C_2. Since $s(1) = 2$ and $s(2) = 2$, we have

$$s(1) = \ln 1 + C_1 + C_2 = 2 \quad \text{and} \quad s(2) = \ln 2 + 2C_1 + C_2 = 2.$$

Solving the system of linear equations

$$\begin{cases} C_1 + C_2 = 2 \\ 2C_1 + C_2 = 2 - \ln 2 \end{cases}$$

We see that $C_1 = -\ln 2$ and $C_2 = 2 + \ln 2$. Thus $s(t) = \ln|t| - (\ln 2)t + 2 + \ln 2.$

19. (10 points) Find $\lim_{x\to 0^-} \Big(1+\sin(3x)\Big)^{\csc(4x)}$

Sol. Take the natural log of $y = (1+\sin\ 3x)^{\csc 4x}$ to have

$$\ln y = \ln\left(1+\sin\ 3x\right)^{\csc 4x} = (\csc 4x)\ln\left(1+\sin\ 3x\right).$$

Consider the limit

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\left\{(\csc 4x)\ln\left(1+\sin\ 3x\right)\right\} = \lim_{x\to 0}\frac{\ln\left(1+\sin\ 3x\right)}{\sin 4x}.$$

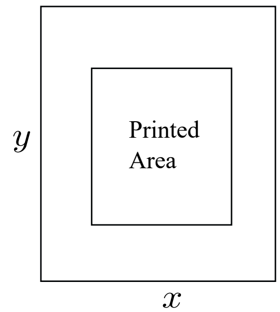
Since the limit is indeterminate form of $\frac{0}{0}$ as $x \to 0$, we can apply L'Hospital's Rule.

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\left(1+\sin\ 3x\right)}{\sin 4x} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{3\cos 3x}{1+\sin 3x}}{4\cos 4x} = \frac{\frac{3\cdot\cos 0}{1+\sin 0}}{4\cos 0} = \frac{\frac{3}{1}}{4} = \frac{3}{4}.$$

Since the exponential function e^x is continuous, the limit of composition becomes

$$\lim_{x\to 0}\left(1+\sin\ 3x\right)^{\csc 4x} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\left(\lim_{x\to 0}\ln y\right)} = e^{\frac{3}{4}}$$

20. (12 points) A poster is to have a printed region with a 4 inch border on the top and bottom and a 3 inch border on the sides. The total area of the whole poster must be 108 in^2. Find the dimensions of the whole poster that would maximize the area of the printed region. Be sure to show that your answer is a maximum.



Sol. Let x and y be the sides as in the figure above. Then the area of the whole poster must be $xy = 108$ or $y = \frac{108}{x}$. The printed area becomes

$$(x-6)(y-8) = (x-6)\left(\frac{108}{x} - 8\right) = 108 - 8x - \frac{6 \cdot 108}{x} + 48 = f(x)$$

The above area function has the derivative

$$f'(x) = -8 + \frac{6 \cdot 108}{x^2}.$$

So $f'(x) = 0$ if $x = 9$ because

$$-8 + \frac{6 \cdot 108}{x^2} = 0 \quad \Leftrightarrow \quad 8 = \frac{6 \cdot 108}{x^2} \quad \Leftrightarrow \quad x^2 = \frac{6 \cdot 108}{8} = 81.$$

We claim that the area function $f(x)$ attains the maximum at $x = 10$. Since

$$f''(x) = \frac{-2 \cdot 6 \cdot 108}{x^3} < 0 \text{ for all } x > 0$$

the area function $f(x)$ is concave downward, and so the critical number $x = 10$ determines the only local maximum, and hence the absolute maximum of $f(x)$ by the 2^{nd} Derivative Test.

We can also use the 1^{st} Derivative Test. One can check that $f'(x)$ changes its sign at $x = 9$. For example,

$$f'(6) = -8 + \frac{6 \cdot 108}{6^2} = -8 + 18 > 0 \text{ and } f'(12) = -8 + \frac{6 \cdot 108}{12^2} = -8 + \frac{9}{2} < 0.$$

The desired dimensions of the whole poster is

$$x = 9 \text{ and } y = \frac{108}{9} = 12.$$