MATH151, Fall 2022
Common Exam I - Version B Solutions

LAST NAME (print): ________________________ FIRST NAME (print): ________________________

INSTRUCTOR: ______________________

SECTION NUMBER: ____________

DIRECTIONS:

- No calculators, cell phones, smart watches, headphones, or other electronic devices may be used, and must be put away.
- TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
- In Part I, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
- In Part II, present your solutions in the space provided. Show all your work neatly and concisely, and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
- **Be sure to fill in your name, UIN, section number, and version letter of the exam on the ScanTron form.**

THE AGGIE HONOR CODE

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ____

Part I: Multiple Choice. 3 points each

1. Find the cosine of the angle between the vectors $\langle -3, 5\rangle$ and $\langle -2, 1\rangle$.

   (a) $\dfrac{11}{\sqrt{34}\sqrt{5}}$ ← correct

   (b) $\dfrac{11}{\sqrt{2}}$

   (c) $\dfrac{11}{4\sqrt{5}}$

   (d) $-\dfrac{1}{\sqrt{34}\sqrt{5}}$

   (e) $\dfrac{30}{\sqrt{34}\sqrt{5}}$

   Solution. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{6+5}{\sqrt{34}\sqrt{5}} = \dfrac{11}{\sqrt{34}\sqrt{5}}$

2. A crate is hauled 10 meters up a ramp under a constant force of 7 Newtons applied at an angle of 30° to the ramp. Find the work done.

   (a) $35\sqrt{2}$ J

   (b) 35 J

   (c) 70 J

   (d) $35\sqrt{3}$ J ← correct

   (e) None of these.

   Solution. $W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta = (7)(10)\cos(30^\circ) = 70(\sqrt{3}/2) = 35\sqrt{3}$

3. Find $\displaystyle\lim_{x\to\infty}\left[\ln(5x^2+1) - \ln(x^3+x-4)\right]$.

   (a) ∞

   (b) $-\infty$ ← correct

   (c) 0

   (d) $\ln(5)$

   (e) $\dfrac{2}{3}$

   Solution. $\displaystyle\lim_{x\to\infty}\left[\ln(5x^2+1) - \ln(x^3+x-4)\right] = \lim_{x\to\infty}\ln\left(\frac{5x^2+1}{x^3+x-4}\right)$, where

$$\lim_{x\to\infty}\frac{5x^2+1}{x^3+x-4} = \lim_{x\to\infty}\frac{5x^2}{x^3} = \lim_{x\to\infty}\frac{5}{x} = 0^+$$

   so $\displaystyle\lim_{x\to\infty}\ln\left(\frac{5x^2+1}{x^3+x-4}\right) = -\infty$, where $\dfrac{5x^2+1}{x^3+x-4} \to 0^+$.

4. Find the value of a that makes $f(x) = \begin{cases} ax^2 - 2x & \text{if } x \le 2 \\ 3x + a & \text{if } x > 2 \end{cases}$ continuous everywhere.

(a) 2
(b) $\frac{10}{3}$ ← correct
(c) $\frac{2}{3}$
(d) 4
(e) There is no such a value.

Solution. We need $\lim_{x\to 2} f(x) = f(2)$:

$$\begin{aligned} f(2) &= 4a - 4 \\ \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (ax^2 - 2x) = 4a - 4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (3x + a) = 6 + a \\ \implies & 4a - 4 = 6 + a \\ \implies & a = 10/3 \end{aligned}$$

5. The position (in meters) of a particle after t seconds is given by $f(t) = (t+1)^2 - 1$. Find the average velocity of the particle from $t = 1$ to $t = 3$.

(a) 8 m/s
(b) 4 m/s
(c) 6 m/s ← correct
(d) $\frac{26}{3}$ m/s
(e) 9 m/s

Solution. $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{15 - 3}{2} = 6$

6. Which of the following is equal to $\tan\left(\arccos\left(\frac{x}{2}\right)\right)$?

(a) $\dfrac{\sqrt{4 + x^2}}{x}$
(b) $\dfrac{\sqrt{4 - x^2}}{x}$ ← correct
(c) $\dfrac{\sqrt{4 - x^2}}{2}$
(d) $\dfrac{x}{\sqrt{4 - x^2}}$
(e) $\dfrac{2}{\sqrt{4 - x^2}}$

Solution. Let $\theta = \arccos\left(\frac{x}{2}\right) \implies \cos\theta = \frac{x}{2} \left[\frac{Adj}{Hyp}\right]$. This gives a reference triangle with $Opp = \sqrt{4 - x^2}$, so $\tan\left(\arccos\left(\frac{x}{2}\right)\right) = \tan\theta = \dfrac{\sqrt{4 - x^2}}{x} \left[\dfrac{Opp}{Adj}\right]$.

7. Find the scalar projection of $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$.

(a) $-\dfrac{4}{\sqrt{13}}$

(b) $\dfrac{2}{\sqrt{34}}$

(c) $-\dfrac{4}{\sqrt{29}}$ ← correct

(d) $\dfrac{4}{\sqrt{29}}$

(e) $\dfrac{4}{\sqrt{13}}$

Solution. $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{6 - 10}{\sqrt{4 + 25}} = \dfrac{-4}{\sqrt{29}}$

8. Find $\displaystyle\lim_{x \to -\infty} \frac{2x + 3}{\sqrt{9x^2 - 2x + 5}}$.

(a) $\dfrac{2}{3}$

(b) $-\dfrac{2}{9}$

(c) $\dfrac{2}{9}$

(d) $-\dfrac{2}{3}$ ← correct

(e) 0

Solution.

$$\lim_{x \to -\infty} \frac{2x + 3}{\sqrt{9x^2 - 2x + 5}} = \lim_{x \to -\infty} \frac{2x}{\sqrt{9x^2}} = \lim_{x \to -\infty} \frac{2x}{3|x|} = \lim_{x \to -\infty} \frac{2x}{3(-x)} = -\frac{2}{3}$$

9. Find a vector equation of the line that passes through $(-6, 4)$ and is perpendicular to the line with parametric equations $x = 7 + 2t$, $y = 1 - 3t$.

(a) $\mathbf{r}(t) = \langle 4 + 3t,\ -6 + 2t \rangle$

(b) $\mathbf{r}(t) = \langle -6 - t,\ 4 + 7t \rangle$

(c) $\mathbf{r}(t) = \langle -6 + 2t,\ 4 - 3t \rangle$

(d) $\mathbf{r}(t) = \langle 4 + 2t, -6 - 3t \rangle$

(e) $\mathbf{r}(t) = \langle -6 + 3t,\ 4 + 2t \rangle$ ← correct

Solution. We have $\mathbf{r}_0 = \langle -6, 4 \rangle$ and $\mathbf{v} = \langle 2, -3 \rangle^{\perp} = \langle 3, 2 \rangle$, so $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle -6 + 3t,\ 4 + 2t \rangle$.

10. Find a vector of length 3 in the same direction as the vector from the point $(-2, 3)$ to $(1, -1)$.

(a) $\left\langle \frac{9}{5}, -\frac{12}{5} \right\rangle$ ← correct

(b) $\left\langle -\frac{9}{5}, \frac{12}{5} \right\rangle$

(c) $\left\langle \frac{9}{\sqrt{5}}, -\frac{12}{\sqrt{5}} \right\rangle$

(d) $\left\langle -\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}} \right\rangle$

(e) $\left\langle -\frac{3}{\sqrt{17}}, -\frac{12}{\sqrt{17}} \right\rangle$

Solution. Consider $A(-2, 3)$ and $B(1, -1)$:

$$\overrightarrow{AB} = \langle 1-(-2), -1-3 \rangle = \langle 3, -4 \rangle \implies \mathbf{u} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{\langle 3, -4 \rangle}{\sqrt{9+16}} = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle \implies 3\mathbf{u} = \left\langle \frac{9}{5}, \frac{-12}{5} \right\rangle$$

11. According to the Intermediate Value Theorem, which of the following intervals contains a solution to the equation $x^3 - 4x - 5 = 0$?

(a) $(-2, -1)$

(b) $(0, 1)$

(c) $(-1, 0)$

(d) $(2, 3)$ ← correct

(e) $(1, 2)$

Solution. We have $f(2) < 0 < f(3)$. (Note that $f(-2) = -5$, $f(-1) = -2$, $f(0) = -5$, $f(1) = -8$, $f(2) = -5$, $f(3) = 10$)

12. Find $\lim_{x \to 2^-} \frac{3x+6}{x-2}$.

(a) ∞

(b) 0

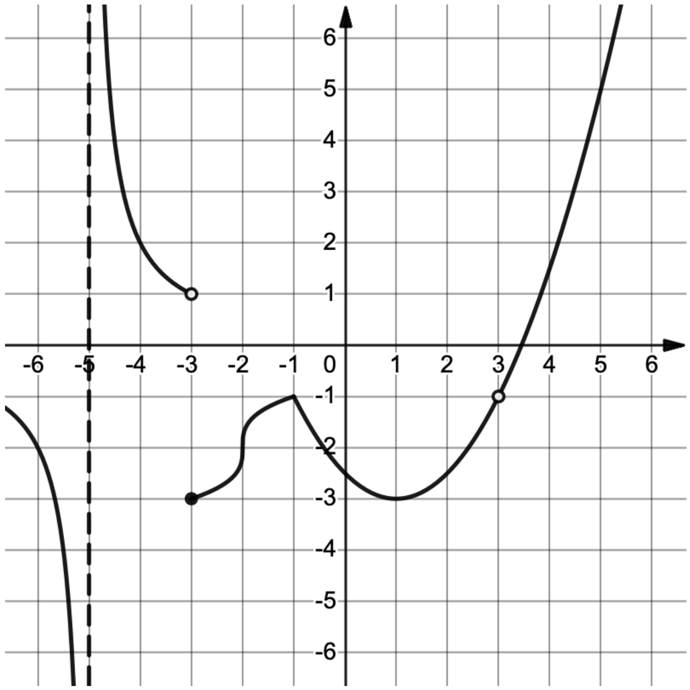
(c) 3

(d) 12

(e) $-\infty$ ← correct

Solution. Note that $x < 2$, so $x - 2 < 0$: $\lim_{x \to 2^-} \frac{3x+6}{x-2} = \text{“}\frac{12}{0^-}\text{”} = -\infty$

Use the following graph of f to answer questions 13 and 14.



13. Given the graph of f above, which of the following is **false**?

    (a) $\lim_{x \to 5} f(x) = 5$.

    (b) $\lim_{x \to 3} f(x) = -1$.

    (c) f is continuous from the left at $x = -3$. $\leftarrow$ correct

    (d) $\lim_{x \to -5^+} f(x) = \infty$.

    (e) f is continuous at $x = -1$.

    Solution. $\lim_{x \to -3^-} f(x) = 1$, but $f(-3) = -3$

14. Given the graph of f above, for which value of x is $f(x)$ differentiable?

    (a) $x = -3$

    (b) $x = 3$

    (c) $x = -1$

    (d) $x = 1$ $\leftarrow$ correct

    (e) $x = -2$

    Solution. f has a horizontal tangent line at $x = 1$, so $f'(1) = 0$. (discontinuities at $x = -3, 3$; corner at $x = -1$; vertical tangent at $x = -2$)

15. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{x^2-4}{(2-x)(x+3)}$.

(a) $y = -1$, $x = 2$, $x = -3$

(b) $y = 1$, $x = -3$

(c) $y = -1$, $x = -3$ ← correct

(d) $y = 1$, $x = 2$

(e) $y = -1$, $y = 1$, $x = -3$

Solution. For horizontal asymptotes, we want x-values that give infinite limits (nonzero/0):

$$\frac{x^2-4}{(2-x)(x+3)} = \frac{(x+2)(x-2)}{-(x-2)(x+3)} \to \frac{(x+2)}{-(x+3)}$$

Thus $x = -3$ is a horizontal asymptote (there is a hole at $x = 2$). For vertical asymptotes, we need y-values corresponding to limits at infinity:

$$\frac{x^2-4}{(2-x)(x+3)} = \frac{x^2-4}{-x^2-x+6} \overset{x\to\pm\infty}{\Longrightarrow} = \frac{x^2}{-x^2} = -1$$

Thus $y = -1$ is a vertical asymptote.

16. Find $\displaystyle\lim_{x\to-\infty} \frac{5e^{-2x}+3e^{4x}}{2e^{-2x}-e^{4x}}$.

(a) -3

(b) 0

(c) ∞

(d) $\dfrac{5}{2}$ ← correct

(e) $-\infty$

Solution. $\displaystyle\lim_{x\to-\infty} \frac{5e^{-2x}+3e^{4x}}{2e^{-2x}-e^{4x}} = \lim_{x\to-\infty} \frac{5e^{-2x}}{2e^{-2x}} = \frac{5}{2}$ (Note that $\displaystyle\lim_{x\to-\infty} e^{-2x} = \infty$ and $\displaystyle\lim_{x\to-\infty} e^{4x} = 0$)

17. Which of the following correctly describes the curve given by $\mathbf{r}(t) = \langle 1+\sin t, -2+\cos t\rangle$ as t increases?

(a) Clockwise around a circle with center $(1, -2)$. ← correct

(b) Counterclockwise around a circle with center $(1, -2)$.

(c) Clockwise around a circle with center $(-1, 2)$.

(d) Counterclockwise around a circle with center $(-1, 2)$.

(e) None of these.

Solution. $x = 1 + \sin t \implies \sin t = x - 1$ and $y = -2 + \cos t \implies \cos t = y + 2$.

$$\sin^2 t + \cos^2 t = 1 \implies (x-1)^2 + (y+2)^2 = 1$$

This is a circle of radius 1 with center $(1, -2)$. We can plot some points to see clockwise behavior:

t	x	y
0	1	-1
$\pi/2$	2	-2
π	1	-3

18. Find $\lim_{x\to\infty} \arctan\left(\frac{x-x^3}{x+7}\right)$.

(a) $\frac{\pi}{2}$

(b) $-\frac{\pi}{4}$

(c) $\frac{\pi}{4}$

(d) $-\infty$

(e) $-\frac{\pi}{2}$ ← correct

Solution. $\lim_{x\to\infty} \frac{x-x^3}{x+7} = \lim_{x\to\infty} \frac{-x^3}{x} = \lim_{x\to\infty} -x^2 = -\infty \implies \lim_{x\to\infty} \arctan\left(\overbrace{\frac{x-x^3}{x+7}}^{-\infty}\right) = -\frac{\pi}{2}$

19. Given $f(x) = x^3 - 4x + 1$ and $f'(x) = 3x^2 - 4$, find the equation of the tangent line to $f(x)$ at $x = -2$.

(a) $y = 8x - 10$

(b) $y = 8x + 1$

(c) $y = x + 10$

(d) $y = 8x + 17$ ← correct

(e) $y = x + 3$

Solution. $f(-2) = -8 + 8 + 1 = 1, f'(-2) = 12 - 4 = 8 \implies y - 1 = 8(x+2) \implies y = 8x + 17$

20. Find $\lim_{x\to\infty} \frac{4x - \sqrt{x^2+5x}}{3x}$.

(a) $\frac{4}{3}$

(b) $\frac{5}{3}$

(c) 1 ← correct

(d) ∞

(e) $-\infty$

Solution. $\lim_{x\to\infty} \frac{4x - \sqrt{x^2+5x}}{3x} = \lim_{x\to\infty} \frac{4x - |x|}{3x} = \lim_{x\to\infty} \frac{4x - x}{3x} = \lim_{x\to\infty} \frac{3x}{3x} = 1$

Part II: Work Out Problems

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

21. (10 points) Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on an object. The force $\mathbf{F}_1$ has a magnitude of 20 lbs and a direction of 45° counterclockwise from the positive x-axis, and $\mathbf{F}_2$ has a magnitude of 6 lbs and a direction of 150° counterclockwise from the positive x-axis.

(a) Find the resultant force $\mathbf{F}$.

Solution.

$$\mathbf{F}_1 = |\mathbf{F}_1|\langle \cos(45°), \sin(45°)\rangle = 20\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \left\langle 10\sqrt{2}, 10\sqrt{2}\right\rangle$$

$$\mathbf{F}_2 = |\mathbf{F}_2|\langle \cos(150°), \sin(150°)\rangle \qquad \Big(= |\mathbf{F}_2|\langle -\cos(30°), \sin(30°)\rangle\Big)$$
$$= 6\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$$
$$= \left\langle -3\sqrt{3}, 3\right\rangle$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \left\langle 10\sqrt{2} - 3\sqrt{3}, 10\sqrt{2} + 3\right\rangle$$

(b) Find the resultant angle θ as measured counterclockwise from the positive x-axis. Leave your answer in terms of an inverse trigonometric expression.

Solution.

$$\tan\theta = \frac{10\sqrt{2}+3}{10\sqrt{2}-3\sqrt{3}} \implies \theta = \boxed{\tan^{-1}\left(\frac{10\sqrt{2}+3}{10\sqrt{2}-3\sqrt{3}}\right)}$$

22. (15 points) Find the following limits. *Do not use L'Hospital's Rule.*

(a) $\lim_{x\to 8} \frac{(x-3)^2-25}{x^2-8x}$

Solution.

$$\lim_{x\to 8} \frac{(x-3)^2-25}{x^2-8x} = \lim_{x\to 8} \frac{x^2-6x+9-25}{x(x-8)} = \lim_{x\to 8} \frac{x^2-6x-16}{x(x-8)} = \lim_{x\to 8} \frac{(x-8)(x+2)}{x(x-8)}$$

$$= \lim_{x\to 8} \frac{x+2}{x}$$

$$= \frac{10}{8} = \boxed{\frac{5}{4}}$$

(b) $\lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2}$

Solution.

$$\lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} = \lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} \cdot \frac{\sqrt{2x+5}+3}{\sqrt{2x+5}+3}$$

$$= \lim_{x\to 2} \frac{(2x+5)-9}{(x-2)(\sqrt{2x+5}+3)}$$

$$= \lim_{x\to 2} \frac{2x-4}{(x-2)(\sqrt{2x+5}+3)}$$

$$= \lim_{x\to 2} \frac{2(x-2)}{(x-2)(\sqrt{2x+5}+3)}$$

$$= \lim_{x\to 2} \frac{2}{\sqrt{2x+5}+3}$$

$$= \frac{2}{6} = \boxed{\frac{1}{3}}$$

(c) $\lim_{x\to -1^-} \frac{|x+1|}{x^2+6x+5}$

Solution. Since $x < -1$, $|x+1| = -(x+1)$:

$$\lim_{x\to -1^-} \frac{|x+1|}{x^2+6x+5} = \lim_{x\to -1^-} \frac{-(x+1)}{(x+1)(x+5)} = \lim_{x\to -1^-} \frac{-1}{x+5} = \frac{-1}{-1+5} = \boxed{-\frac{1}{4}}$$

23. (6 points) Consider the function $f(x) = \begin{cases} 4e^{x+2} & \text{if } x \le -2 \\ 3x+1 & \text{if } -2 < x < 1 \\ \sqrt{x^2+15} & \text{if } x > 1 \end{cases}$

(a) Find $\lim_{x \to -2} f(x)$, or explain why it does not exist.

Solution.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} 4e^{x+2} = 4e^0 = 4$$

and

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (3x+1) = -6 + 1 = -5$$

$\boxed{\text{These are not equal, so } \lim_{x \to -2} f(x) \text{ does not exist (DNE)}}$.

(b) Find $\lim_{x \to 1^+} f(x)$, or explain why it does not exist.

Solution.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x^2+15} = \sqrt{1+15} = \boxed{4}$$

24. (9 points) Use the **definition** of the derivative to find $f'(x)$ for $f(x) = \dfrac{2}{x-4}$. *No points will be given for any shortcut formulas used!*

Solution.

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\dfrac{2}{(x+h)-4} - \dfrac{2}{x-4}}{h} \cdot \frac{(x+h-4)(x-4)}{(x+h-4)(x-4)} \\ &= \lim_{h \to 0} \frac{2(x-4) - 2(x+h-4)}{h(x+h-4)(x-4)} \\ &= \lim_{h \to 0} \frac{2x - 8 - 2x - 2h + 8}{h(x+h-4)(x-4)} \\ &= \lim_{h \to 0} \frac{-2h}{h(x+h-4)(x-4)} \\ &= \lim_{h \to 0} \frac{-2}{(x+h-4)(x-4)} \\ &= \boxed{\frac{-2}{(x-4)^2}} \end{aligned}$$

Do not write in this table.

Question	Points Awarded	Points
1-20		60
21		10
22		15
23		6
24		9
Total		100