MATH151, Fall 2022
Common Exam II - Version A

LAST NAME (print): _____ FIRST NAME (print): _____

INSTRUCTOR: _____ UIN: _____

SECTION NUMBER: _____

DIRECTIONS:

- No calculators, cell phones, smart watches, headphones, or other electronic devices may be used, and must be put away.

- TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

- In Part I, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*

- In Part II, present your solutions in the space provided. Show all your work neatly and concisely, and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

- **Be sure to fill in your name, UIN, section number, and version letter of the exam on the ScanTron form.**

THE AGGIE HONOR CODE

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: _____

Part I: Multiple Choice. 3.5 points each

1. Find the slope of the tangent line to the curve $3y - xy^2 = 18$ at the point $(-3, 2)$.

(a) $\dfrac{22}{15}$

(b) $-\dfrac{4}{9}$

(c) $-\dfrac{22}{9}$

(d) $-\dfrac{8}{3}$

(e) $\dfrac{4}{15}$ $\leftarrow$ correct

Solution. Implicit differentiation gives

$$3\frac{dy}{dx} - \left[y^2 + x\left(2y \cdot \frac{dy}{dx}\right)\right] = 0$$

$$3\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$

$$3\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{3 - 2xy}$$

$$\implies \frac{dy}{dx}\bigg|_{(-3,2)} = \frac{4}{3 + 12} = \frac{4}{15}$$

2. Find a tangent vector to the curve $\mathbf{r}(t) = \langle \sqrt{6 - t},\ te^{3t}\rangle$ at the point where $t = 2$.

(a) $\left\langle -\dfrac{1}{4},\ 7e^6\right\rangle$ $\leftarrow$ correct

(b) $\left\langle -\dfrac{1}{4},\ 3e^6\right\rangle$

(c) $\left\langle \dfrac{1}{4},\ 7e^6\right\rangle$

(d) $\left\langle \dfrac{1}{4},\ 3e^6\right\rangle$

(e) $\langle -1,\ e^6\rangle$

Solution.

$$\mathbf{r}'(t) = \left\langle \frac{1}{2}(6 - t)^{-1/2} \cdot -1,\ e^{3t} + t \cdot e^{3t} \cdot 3\right\rangle = \left\langle -\frac{1}{2\sqrt{6-t}},\ e^{3t} + 3te^{3t}\right\rangle$$

$$\implies \mathbf{r}'(2) = \left\langle -\frac{1}{2\cdot 2},\ e^6 + 6e^6\right\rangle$$

$$= \left\langle -\frac{1}{4},\ 7e^6\right\rangle$$

3. Find the 123rd derivative of $f(x) = \frac{1}{3}\cos(3x)$.

(a) $3^{122}\cos(3x)$

(b) $-3^{122}\sin(3x)$

(c) $3^{122}\sin(3x)$ $\leftarrow$ correct

(d) $-3^{123}\cos(3x)$

(e) $3^{123}\sin(3x)$

Solution.

$$f(x) = \frac{1}{3}\cos(3x)$$

$$f'(x) = \frac{1}{3}(-3\sin(3x))$$

$$f''(x) = \frac{1}{3}(-3^2\cos(3x))$$

$$f'''(x) = \frac{1}{3}(3^3\sin(3x))$$

The 123rd derivative has a factor of $\dfrac{1}{3}3^{123} = 3^{122}$ and the sign/trig function depends on the remainder of $\dfrac{123}{4} = 30\,r3$, so $f^{(123)}(x) = 3^{122}\sin(3x)$.

4. There are two lines tangent to the parabola $f(x) = x^2 + 1$ that pass through the point $(1, -2)$. Find the x-coordinates where these tangent lines touch the parabola.

(a) $x = -1, 3$ ← correct

(b) $x = 0, 2$

(c) $x = -2, 4$

(d) $x = -\frac{1}{2}, \frac{5}{2}$

(e) None of these.

Solution. Let $(x, x^2 + 1)$ be a point of tangency, then we can calculate the slope of a tangent line two ways:

$$\frac{\Delta y}{\Delta x} = f'(x)$$

$$\frac{(x^2 + 1) - (-2)}{x - 1} = 2x$$

$$x^2 + 3 = 2x(x - 1)$$

$$x^2 + 3 = 2x^2 - 2x$$

$$0 = x^2 - 2x - 3$$

$$0 = (x + 1)(x - 3) \implies x = -1, 3$$

5. Which of the following is true?

$$f(x) = \begin{cases} 2x^3 + 5 & \text{if } x < 1 \\ 3x^2 + 4 & \text{if } 1 \le x \le 3 \\ 10x + 1 & \text{if } x > 3 \end{cases}$$

(a) f is continuous but not differentiable at $x = 1$; f is differentiable at $x = 3$

(b) f is continuous but not differentiable at both $x = 1$ and $x = 3$

(c) f is differentiable at both $x = 1$ and $x = 3$

(d) f is differentiable at $x = 1$; f is continuous but not differentiable at $x = 3$ ← correct

(e) f is discontinuous at both $x = 1$ and $x = 3$

Solution. Continuity:
$x = 1$: $2(1)^3 + 5 = 7$ and $3(1)^2 + 4 = 7$ ✓
$x = 3$: $3(3)^2 + 4 = 31$ and $10(3) + 1 = 31$ ✓

Differentiable: $f'(x) = \begin{cases} 6x^2 & \text{if } x < 1 \\ 6x & \text{if } 1 < x < 3 \\ 10 & \text{if } x > 3 \end{cases}$

$x = 1$: $6(1)^2 = 6$ and $6(1) = 6$ ✓
$x = 3$: $6(3) = 18 \ne 10$ X

6. Find the slope of the tangent line to the curve $x = 6t^2 - 26$, $y = t^2 + 3t + 2$ at the point $(-2, 0)$.

(a) $\dfrac{7}{24}$

(b) $\dfrac{1}{24}$ ← correct

(c) $-\dfrac{1}{24}$

(d) $\dfrac{24}{7}$

(e) 24

Solution. The tangent slope is given by $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+3}{12t}$. We next find the t-value for $(-2, 0)$:

$$6t^2 - 26 = -2 \qquad\qquad t^2 + 3t + 2 = 0$$
$$6t^2 - 24 = 0 \qquad\qquad (t+1)(t+2) = 0$$
$$6(t^2 - 4) = 0 \qquad\qquad t = \cancel{-1}, -2$$
$$6(t+2)(t-2) = 0$$
$$t = -2, \cancel{2}$$

$$\implies \left.\frac{dy}{dx}\right|_{t=-2} = \frac{1}{24}.$$

7. Find the linearization of $f(x) = x^{3/2}$ at $a = 4$ and use it to approximate $f(4.1)$.

(a) 7.9

(b) 8.1

(c) 8.2

(d) 8.3 ← correct

(e) 8.4

Solution.

$$L(x) = f(4) + f'(4)(x - 4)$$
$$f(4) = 8$$
$$f'(x) = \frac{3}{2}x^{1/2} \implies f'(4) = 3$$

$$\implies L(x) = 8 + 3(x - 4)$$
$$L(4.1) = 8 + 3(0.1) = 8.3$$

8. Find the t-value(s) where the curve $x = \sin^2 t + \cos t$, $y = t - \cos t$, $0 \le t < 2\pi$ has a vertical tangent line.

(a) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

(b) $0, \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \pi$ ← correct

(c) $0, \dfrac{\pi}{6}, \dfrac{11\pi}{6}, \pi$

(d) $\dfrac{3\pi}{2}$

(e) $0, \dfrac{\pi}{3}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}, \pi$

Solution.

$$\frac{dx}{dt} = 0$$
$$2\sin t \cos t - \sin t = 0$$
$$\sin t(2\cos t - 1) = 0$$

$$\sin t = 0 \qquad\qquad 2\cos t - 1 = 0$$
$$t = 0, \pi \qquad\qquad \cos t = \frac{1}{2}$$
$$\qquad\qquad t = \frac{\pi}{3}, \frac{5\pi}{3}$$

4

9. The position of a particle moving in a straight line is given by $s(t) = \frac{2}{3}t^3 - t^2 + 5t$, where s is in feet and t is in seconds. Find the velocity at the time where the acceleration of the particle is 10 ft/s².

(a) 3 ft/s

(b) 9 ft/s

(c) 17 ft/s ← correct

(d) 24 ft/s

(e) 5 ft/s

Solution. $v(t) = s'(t) = 2t^2 - 2t + 5$ and $a(t) = v'(t) = 4t - 2$. Solving $a(t) = 10$ gives

$$4t - 2 = 10 \implies 4t = 12 \implies t = 3$$

and $v(3) = 18 - 6 + 5 = 17$.

10. Find the slope of the tangent line to $f(x) = \arctan(2x)$ at $x = 3$.

(a) $\dfrac{2}{37}$ ← correct

(b) $\dfrac{2}{19}$

(c) $\dfrac{1}{37}$

(d) $\dfrac{1}{19}$

(e) $\dfrac{1}{7}$

Solution. $f'(x) = \dfrac{1}{1 + (2x)^2} \cdot 2 = \dfrac{2}{1 + 4x^2} \implies f'(3) = \dfrac{2}{37}$

11. A bacteria culture grows at a rate proportional to its size. It has 100 bacteria initially and 400 bacteria after 3 hours. How long will it take for there to be 1000 bacteria?

(a) $\dfrac{4\ln(10)}{\ln(3)}$ hours

(b) $\dfrac{\ln(10)}{3\ln(4)}$ hours

(c) $\dfrac{30}{\ln(4)}$ hours

(d) $\dfrac{3\ln(10)}{\ln(4)}$ hours ← correct

(e) None of these.

Solution. We have $y(t) = y_0 e^{kt} = 100 e^{kt}$ and $y(3) = 400$. Thus

$$100e^{3k} = 400$$
$$e^{3k} = 4$$
$$3k = \ln(4)$$
$$k = \frac{\ln(4)}{3}$$

and $y(t) = 100 \cdot 4^{t/3}$. We next solve $y(t) = 1000$:

$$100 \cdot 4^{t/3} = 1000$$
$$4^{t/3} = 10$$
$$\frac{t}{3}\ln(4) = \ln(10)$$
$$t = \frac{3\ln(10)}{\ln(4)}$$

12. An object is in motion according to $s(t) = t^2 - 4t + 7$, $t \geq 0$, where t is in seconds and s is in feet. Find the total distance traveled by the object during the first three seconds.

(a) 2 ft

(b) 3 ft

(c) 4 ft

(d) 5 ft ← correct

(e) 7 ft

Solution. $v(t) = 2t - 4 = 0 \implies t = 2$

$$s(0) = 7$$
$$s(2) = 3$$
$$s(3) = 4 \qquad \text{so the total distance is } |3 - 7| + |4 - 3| = 5$$

13. The radius of a sphere was measured to be 10 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated surface area of the sphere. (The surface area of a sphere is $S = 4\pi r^2$.)

(a) 4π cm^2

(b) 16π cm^2 ← correct

(c) 8π cm^2

(d) 40π cm^2

(e) 80π cm^2

Solution. $S = 4\pi r^2 \implies dS = 8\pi r \, dr = 8\pi(10)(0.2) = 16\pi$

14. Find the derivative of $f(x) = \ln\left(\dfrac{\csc^3(x)}{\sqrt{x^4 + 5}}\right)$.

(a) $3\csc(x) - \dfrac{1}{8x^3}$

(b) $-3\cot(x) - \dfrac{4x^3}{x^4 + 5}$

(c) $-3\cot(x) - \dfrac{2x^3}{x^4 + 5}$ ← correct

(d) $-\dfrac{3}{\sec(x)\tan(x)} - \dfrac{2x^3}{x^4 + 5}$

(e) $-3\csc^2(x)\cot(x) - \dfrac{4x^3}{\sqrt{x^4 + 5}}$

Solution.

$$f(x) = \ln\left(\frac{\csc^3(x)}{\sqrt{x^4 + 5}}\right) = \ln(\csc^3(x)) - \ln(x^4 + 5)^{1/2}$$

$$= 3\ln(\csc(x)) - \frac{1}{2}\ln(x^4 + 5)$$

$$\implies f'(x) = 3 \cdot \frac{-\csc(x)\cot(x)}{\csc(x)} - \frac{1}{2}\frac{4x^3}{x^4 + 5}$$

$$= -3\cot(x) - \frac{2x^3}{x^4 + 5}$$

6

Suppose f and g are differentiable functions. Use the following table of values to answer questions 15 and 16.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	3	4	2	1
1	-1	2	4	-5

15. Find $H'(1)$ if $H(x) = \dfrac{g(f(x))}{2x+1}$.

(a) $\dfrac{20}{9}$

(b) $\dfrac{4}{3}$

(c) $-\dfrac{64}{9}$

(d) $-\dfrac{2}{9}$

(e) $\dfrac{4}{9}$ ← correct

Solution.

$$H'(x) = \frac{(2x+1)[g'(f(x))f'(x)] - g(f(x))(2)}{(2x+1)^2} \implies H'(1) = \frac{(3)[g'(f(1))f'(1)] - g(f(1))(2)}{3^2}$$

$$= \frac{(3)[g'(-1)(4)] - g(-1)(2)}{9}$$

$$= \frac{(3)(1)(4) - (4)(2)}{9} = \frac{4}{9}$$

16. Find $h'(-1)$ if $h(x) = [f(x^2)]^2$.

(a) 4

(b) -8

(c) -24

(d) 16 ← correct

(e) Not enough information.

Solution. $h'(x) = 2f(x^2)f'(x^2) \cdot 2x \implies h'(-1) = 2f(1)f'(1) \cdot -2 = 2(-1)(4)(-2) = 16$

Part II: Work Out Problems

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

17. (15 points) Differentiate the following functions. Do not simplify after taking the derivative.

    (a) $y = \sec^3\left(3x^5 + 7^{-2x}\right)$

    Solution.

    $$y' = 3\sec^2(3x^5 + 7^{-2x}) \cdot \sec(3x^5 + 7^{-2x})\tan(3x^5 + 7^{-2x}) \cdot (15x^4 + 7^{-2x}\ln(7)\cdot -2)$$

    (b) $y = \dfrac{\tan(4x)}{(x^3 + e^{\pi})^2}$

    Solution 1.

    $$y' = \frac{(x^3 + e^{\pi})^2(4\sec^2(4x)) - \tan(4x)[2(x^3 + e^{\pi})\cdot 3x^2]}{(x^3 + e^{\pi})^4}$$

    Solution 2.

    $$\ln y = \ln(\tan(4x)) - 2\ln(x^3 + e^{\pi})$$

    $$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{4\sec^2(4x)}{\tan(4x)} - 2\frac{3x^2}{x^3 + e^{\pi}}$$

    $$y = \left[\frac{4\sec^2(4x)}{\tan(4x)} - \frac{6x^2}{x^3 + e^{\pi}}\right]\frac{\tan(4x)}{(x^3 + e^{\pi})^2}$$

    (c) $y = \log_{10}(5 - x^2)e^{\sqrt{x}}$

    Solution 1.

    $$y' = \frac{-2x}{(5 - x^2)\ln(10)}e^{\sqrt{x}} + \log_{10}(5 - x^2)\left[e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}\right]$$

    Solution 2.

    $$\ln y = \ln(\log_{10}(5 - x^2)) + \sqrt{x}$$

    $$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{1}{\log_{10}(5 - x^2)}\cdot\frac{-2x}{(5 - x^2)\ln(10)} + \frac{1}{2}x^{-1/2}$$

    $$y = \left[\frac{1}{\log_{10}(5 - x^2)}\cdot\frac{-2x}{(5 - x^2)\ln(10)} + \frac{1}{2}x^{-1/2}\right]\log_{10}(5 - x^2)e^{\sqrt{x}}$$

18. (8 points) Differentiate $y = (x^5 + 4x^3)^{\arcsin(2x)}$. Your answer should be in terms of x only.

Solution. We apply logarithmic differentiation:

$$\ln y = \ln(x^5 + 4x^3)^{\arcsin(2x)} = \arcsin(2x) \ln(x^5 + 4x^3)$$

$$\xRightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{2}{\sqrt{1-(2x)^2}} \ln(x^5 + 4x^3) + \arcsin(2x) \frac{5x^4 + 12x^2}{x^5 + 4x^3}$$

$$y' = \left[\frac{2}{\sqrt{1-(2x)^2}} \ln(x^5 + 4x^3) + \arcsin(2x) \frac{5x^4 + 12x^2}{x^5 + 4x^3} \right] (x^5 + 4x^3)^{\arcsin(2x)}$$

19. (9 points) Find $\dfrac{dy}{dx}$ for $\sin(xy^2) + e^{y-x} = 9$.

**Solution.** We apply implicit differentiation:

$$\cos(xy^2) \left(y^2 + x \cdot 2y \frac{dy}{dx} \right) + e^{y-x} \left(\frac{dy}{dx} - 1 \right) = 0$$

$$y^2 \cos(xy^2) + 2xy \cos(xy^2) \frac{dy}{dx} + e^{y-x} \frac{dy}{dx} - e^{y-x} = 0$$

$$2xy \cos(xy^2) \frac{dy}{dx} + e^{y-x} \frac{dy}{dx} = e^{y-x} - y^2 \cos(xy^2)$$

$$\frac{dy}{dx} = \frac{e^{y-x} - y^2 \cos(xy^2)}{2xy \cos(xy^2) + e^{y-x}}$$

20. (12 points) A camera is positioned 8 m away from the launching point of a weather balloon. The balloon is released and rises vertically at a speed of 3 m/s.

(a) At what rate is the distance between the camera and the balloon changing when the balloon has risen 6 m?

Solution 1. Let y be the distance the balloon has risen and z be the distance between the camera and balloon.
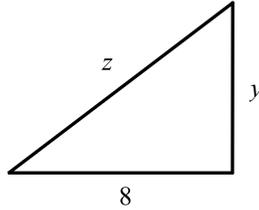
We are given $\dfrac{dy}{dt} = 3$ and want to find $\dfrac{dz}{dt}$ when $y = 6$. We have

$$8^2 + y^2 = z^2 \xRightarrow{\frac{d}{dt}} \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

When $y = 6$, we have $z = \sqrt{8^2 + 6^2} = 10$ and the related rate equation gives

$$(6)(3) = 10\frac{dz}{dt} \implies \frac{dz}{dt} = \frac{18}{10} = \frac{9}{5}$$
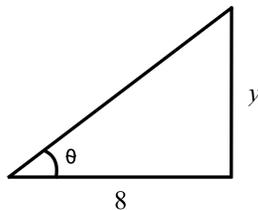
Solution 2. Alternatively, we can use

$$z = \sqrt{64 + y^2} \xRightarrow{\frac{d}{dt}} \frac{dz}{dt} = \frac{1}{\cancel{2}\sqrt{64 + y^2}} \cdot \cancel{2}y\frac{dy}{dt}$$

When $y = 6$, we have

$$\frac{dz}{dt} = \frac{1}{10}(6)(3) = \frac{9}{5}$$

(b) If the camera is always kept aimed at the balloon, how fast is the camera's angle of elevation changing when the balloon has risen 6 m?

Solution 1. Let y be the distance the balloon has risen and θ be the camera's angle of elevation.

We are given $\dfrac{dy}{dt} = 3$ and want $\dfrac{d\theta}{dt}$ when $y = 6$. We have

$$\tan\theta = \frac{y}{8} \overset{\frac{d}{dt}}{\implies} \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{8}\frac{dy}{dt}$$

When $y = 6$, the hypotenuse of the triangle is 10 (see part a). Thus $\sec\theta = \dfrac{10}{8}$ and the related rate equation gives

$$\left(\frac{10}{8}\right)^2\frac{d\theta}{dt} = \frac{1}{8}(3) \implies \frac{d\theta}{dt} = \frac{24}{100} = \frac{6}{25}$$

Solution 2. Alternatively, we can use

$$\theta = \arctan\left(\frac{y}{8}\right) \overset{\frac{d}{dt}}{\implies} \frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{y}{8}\right)^2}\cdot\frac{1}{8}\frac{dy}{dt}$$

When $y = 6$, we have

$$\frac{d\theta}{dt} = \frac{1}{1 + \frac{9}{16}}\cdot\frac{1}{8}(3) = \frac{1}{\frac{25}{16}}\frac{3}{8} = \frac{6}{25}$$

Do not write in this table.

Question	Points Awarded	Points
1–16		56
17		15
18		8
19		9
20		12
Total		100