Exam3 version A

Friday, November 19, 2021 9:10 AM

exam3Akey

MATH 152, FALL 2021
COMMON EXAM III - VERSION A KEY

LAST NAME(print): ____________________ FIRST NAME(print): ____________________

INSTRUCTOR: ____________________

SECTION NUMBER: __________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1 (Problems 1-15), mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2 (Problems 16-19), present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
5. Be sure to *write your name, section number and version letter of the exam on the ScanTron form.*
6. **Again. The use of a calculator, laptop or computer is prohibited.**

THE AGGIE HONOR CODE

"An Aggie does not lie, cheat, or steal, or tolerate those who do."

Signature: ____________________

FOR INSTRUCTOR USE ONLY

Question	Points Awarded	Points
1-15	ScanTron	60
16		12
17		8
18		12
19		8
TOTAL		100

Part 1: Multiple Choice (4 points each)

1. Which of the following is true regarding the series $\sum_{n=1}^{\infty}\frac{5n\cdot 3^n}{4^n}$.

   (a) The Ratio Test limit is $\frac{3}{4}$, so the series converges. *key*
   (b) The Ratio Test limit is $\frac{3}{4}$, so the series diverges.
   (c) The Ratio Test limit is $\frac{15}{4}$, so the series diverges.
   (d) The Ratio Test limit is $\frac{15}{4}$, so the series converges.
   (e) The Ratio Test limit is $\frac{9}{4}$, so the series diverges.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5(n+1)\cdot 3^{\cancel{n+1}\,1}}{4^{\cancel{n+1}\,1}}\cdot\frac{\cancel{4^n}}{5n\cdot\cancel{3^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{15(n+1)}{20n}\right| = \frac{15}{20} = \frac{3}{4} < 1$$

∴ series converges

2. Find the Maclaurin series for the function $f(x) = x^2e^{-x^3}$.

   (a) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{n!}$ *key*
   (b) $\sum_{n=0}^{\infty}\frac{x^{3n+6}}{n!}$
   (c) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+6}}{n!}$
   (d) $\sum_{n=0}^{\infty}\frac{x^{5n}}{n!}$
   (e) $\sum_{n=0}^{\infty}\frac{x^{3n+2}}{n!}$

NOTE

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$f(x) = x^2\cdot e^{(-x^3)} = x^2\cdot\sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!}$$

$$= x^2\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{3n}}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{3n+2}}{n!}$$

3. Find the 3*rd* degree Taylor polynomial, $T_3(x)$, for the function $f(x) = \ln x$ centered at $a = 6$.

   (a) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{36}(x-6)^2 + \frac{1}{108}(x-6)^3$
   (b) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{72}(x-6)^2 + \frac{1}{648}(x-6)^3$ *key*
   (c) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{6}(x-6)^2 + \frac{1}{36}(x-6)^3$
   (d) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{36}(x-6)^2 + \frac{1}{216}(x-6)^3$
   (e) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{6}(x-6)^2 + \frac{1}{12}(x-6)^3$

$$T_3(x) = f(6) + f'(6)(x-6) + \frac{f''(6)}{2!}(x-6)^2 + \frac{f'''(6)}{3!}(x-6)^3$$

$f(6) = \ln 6$
$f'(x) = \frac{1}{x} \Rightarrow f'(6) = \frac{1}{6}$
$f''(x) = -\frac{1}{x^2} \Rightarrow f''(6) = -\frac{1}{36}$
$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(6) = \frac{2}{6^3} = \frac{2}{216}$

Then

$$T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{72}(x-6)^2 + \frac{1}{648}(x-6)^3$$

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4. Find a power series representation for $f(x) = \frac{x}{x+4}$ and its radius of convergence.

   (a) $\sum_{n=0}^{\infty}\frac{x^{n+1}}{4^{n+1}}$, $R = 4$
   (b) $\sum_{n=0}^{\infty}\frac{x^{n+1}}{4^{n+1}}$, $R = \frac{1}{4}$
   (c) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^{n+1}}$, $R = 4$
   (d) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{4^{n+1}}$, $R = \frac{1}{4}$
   (e) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{4^{n+1}}$, $R = 4$ *key*

$$f(x) = \frac{x}{4+x} = \frac{x}{4}\cdot\frac{1}{1+\frac{x}{4}} = \frac{x}{4}\cdot\frac{1}{1-\left(-\frac{x}{4}\right)}$$

$$= \frac{x}{4}\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n = \frac{x}{4}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{4^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{n+1}}{4^{n+1}}$$

and $|r| = \left|-\frac{x}{4}\right| < 1$

∴ $R = 4$

5. Find the interval of convergence of the series $\sum_{n=0}^{\infty}\frac{n!(x+4)^n}{3^n}$.

   (a) $\{0\}$
   (b) $(-\infty,\infty)$
   (c) $(-7,-1)$
   (d) $(-4,4)$
   (e) $\{-4\}$ *key*

Ratio test

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(x+4)^{\cancel{n+1}\,1}}{3^{\cancel{n+1}\,1}}\cdot\frac{\cancel{3^n}}{\cancel{n!}\,\cancel{(x+4)^n}}\right| = \lim_{n\to\infty}\left|\frac{(x+4)\cdot(n+1)}{3}\right|$$

$$= |x+4|\cdot\lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \begin{cases}\infty & \text{otherwise}\\ 0 & \text{if } x=-4\end{cases}$$

∴ I.C $= \{-4\}$

6. Suppose that $0 < a_n < b_n$ for all $n \geq 1$. Which of the following statements is always true?

   (a) If $\sum_{n=1}^{\infty} b_n$ is divergent, then so is $\sum_{n=1}^{\infty} a_n$.
   (b) If $\sum_{n=1}^{\infty} a_n$ is convergent, then so is $\sum_{n=1}^{\infty} b_n$.
   (c) If $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.
   (d) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$. *key*
   (e) If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} b_n = 0$.

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7. Which of the following statments is true for the series $\sum_{n=1}^{\infty}\frac{3+\sin n}{n^5+1}$?

   (a) The series converges since $\frac{3+\sin n}{n^5+1} < \frac{3}{n^5}$ and $\sum_{n=1}^{\infty}\frac{3}{n^5}$ converges.
   (b) The series converges since $\frac{3+\sin n}{n^5+1} > \frac{2}{n^5}$ and $\sum_{n=1}^{\infty}\frac{2}{n^5}$ converges.
   (c) The series diverges since $\frac{3+\sin n}{n^5+1} > \frac{2}{n^5}$ and $\sum_{n=1}^{\infty}\frac{2}{n^5}$ diverges.
   (d) The series converges since $\frac{3+\sin n}{n^5+1} < \frac{4}{n^5}$ and $\sum_{n=1}^{\infty}\frac{4}{n^5}$ converges. *key*
   (e) None of these.

8. The alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+2)^2}$ converges. Use the Alternating Series Estimation Theorem to determine an upper bound on the absolute value of the error in using s_5 to approximate the sum of the series.

   (a) $\frac{1}{8}$
   (b) $\frac{1}{9}$
   (c) $\frac{1}{64}$ *key*
   (d) $\frac{1}{81}$
   (e) $\frac{1}{35}$

A.S.E.T.

$$|\text{Error}| = |s - s_5| < b_6 = \frac{1}{(6+2)^2} = \frac{1}{8^2} = \frac{1}{64}$$

9. Consider the series below, which statement is true regarding the absolute convergence of each series?

$$\text{(I)}\ \sum_{n=1}^{\infty}\frac{(-2)^n}{3^{n+1}} \qquad \text{(II)}\ \sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+3}$$

   (a) (I) converges but not absolutely, (II) converges absolutely.
   (b) Both (I) and (II) converge but not absolutely.
   (c) Both (I) and (II) converges absolutely.
   (d) (I) converges abolutely, (II) diverges.
   (e) (I) converges absolutely, (II) converges but not absolutely. *key*

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10. For which series is the ratio test inconclusive?

   (a) $\sum_{n=2}^{\infty}\frac{(-1)^n}{3^n\sqrt{\ln n}}$
   (b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$ *key*
   (c) $\sum_{n=1}^{\infty} ne^{-n}$
   (d) $\sum_{n=1}^{\infty}\frac{n}{2^n}$
   (e) $\sum_{n=1}^{\infty}\frac{n+2}{n!}$

ratio test

$$: \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^5}\cdot\frac{n^5}{(-1)^n}\right| = \lim_{n\to\infty}\frac{n^5}{(n+1)^5} = 1 \ : \text{inconclusive.}$$

11. Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}}{4^{2n}(2n)!}$

   (a) $\cos(\frac{3}{4})$ *key*
   (b) $3\cos(\frac{3}{4})$
   (c) $\sin(\frac{3}{4})$
   (d) $\cos 3$
   (e) $\sin 3$

NOTE

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 3^{2n}}{4^{2n}\cdot(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\left(\frac{3}{4}\right)^{2n}}{(2n)!}$$

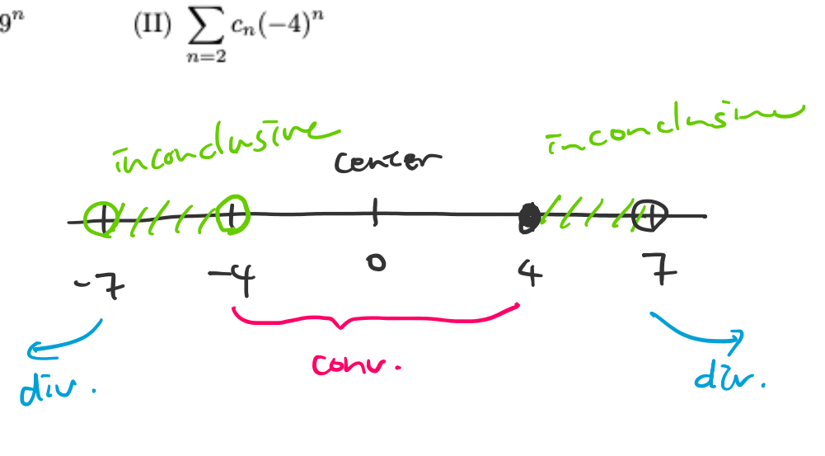
$$= \cos\left(\frac{3}{4}\right)$$

: power series centered at $x=0$

12. The series $\sum_{n=2}^{\infty} c_n x^n$ converges when $x = 4$ and diverges when $x = -7$. What can be said about the convergence of the following series?

$$\text{(I)}\ \sum_{n=2}^{\infty} c_n 9^n \qquad \text{(II)}\ \sum_{n=2}^{\infty} c_n(-4)^n$$

   (a) Both (I) and (II) are inconclusive.
   (b) (I) diverges, (II) converges.
   (c) (I) diverges, (II) is inconclusive. *key*
   (d) Both (I) and (II) converge.
   (e) (I) is inconclusive, (II) converges.



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13. Find the coefficient of x^3 in the Maclaurin series for the function $f(x) = \sin(2x)$.

   (a) $\frac{4}{3}$
   (b) $-\frac{4}{3}$ *key*
   (c) $\frac{2}{3}$
   (d) $-\frac{2}{3}$
   (e) $\frac{1}{3}$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(2x)^{2n+1}}{(2n+1)!}$$

$$\text{if } n=1:\ (-1)^1\cdot\frac{(2x)^3}{3!}$$

$$= -\frac{8x^3}{6} = -\frac{4}{3}x^3$$

14. Which of the following statements is true for the series $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^2+1}$?

   (a) The series converges absolutely.
   (b) The series converges but not absolutely.
   (c) The series diverges by the alternating series test.
   (d) The series diverges by the test for divergence. *key*
   (e) None of these.

15. Evaluate the indefinite integral $\int \arctan(4x^3)\,dx$ as a Maclaurin series.

   (a) $C + \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+2}x^{6n+6}}{(2n+1)(2n+2)}$
   (b) $C + \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+1}x^{6n+6}}{(2n+1)(2n+2)}$
   (c) $C + \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+2}x^{6n+4}}{(2n+1)(6n+4)}$
   (d) $C + \sum_{n=0}^{\infty}\frac{(-1)^n 4^{6n+4}x^{6n+4}}{(2n+1)(6n+4)}$
   (e) $C + \sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+1}x^{6n+4}}{(2n+1)(6n+4)}$ *key*

NOTE

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1}$$

$$\arctan(4x^3) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(4x^3)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}\cdot x^{6n+3}}{2n+1}$$

then

$$\int\arctan(4x^3)\,dx = \int\sum_{n=0}^{\infty}(-1)^n\cdot\frac{4^{2n+1}\cdot x^{6n+3}}{2n+1}\,dx$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\cdot\frac{4^{2n+1}\cdot x^{6n+4}}{(2n+1)\cdot(6n+4)}$$

Part 2: Work Out

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (12 pts) Find (a) the Radius of convergence and (b) Interval of convergence of the power series $\sum_{n=2}^{\infty}\frac{(3x-1)^n}{8^n(n-1)}$.

key: $R = \frac{8}{3}$, $[-\frac{7}{3}, 3)$

Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-1)^{\cancel{n+1}\,1}}{8^{\cancel{n+1}\,1}(n+1-1)}\cdot\frac{\cancel{8^n}\cdot(n-1)}{\cancel{(3x-1)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(3x-1)(n-1)}{8n}\right| = |3x-1|\lim_{n\to\infty}\left|\frac{n-1}{8n}\right|$$

$$= |3x-1|\cdot\frac{1}{8} < 1$$

$$\Rightarrow |3x-1| < 8$$

$$\Rightarrow \left|x-\frac{1}{3}\right| < \frac{8}{3} \quad \therefore R = \frac{8}{3}$$

and $\left(-\frac{7}{3}, 3\right)$

And end points

$$\text{If } \boxed{x=-\frac{7}{3}} \Rightarrow \sum_{n=2}^{\infty}\frac{(-8)^n}{8^n\cdot(n-1)} = \sum_{n=2}^{\infty}\frac{(-1)^n\cdot 8^n}{8^n\cdot(n-1)} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}$$

: converges by A.S.T (or L.C.T with $\sum\frac{(-1)^n}{n}$)

$$\text{If } \cancel{x=3} \Rightarrow \sum_{n=2}^{\infty}\frac{8^n}{8^n(n-1)} = \sum_{n=2}^{\infty}\frac{1}{n-1}$$

: diverges by Comparison Test with $\sum_{n=2}^{\infty}\frac{1}{n}$ (or L.C.T)

$$\therefore R = \frac{8}{3},\ \text{I.C}: \left[-\frac{7}{3}, 3\right)$$

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17. (8 pts) Find the Taylor Series for $f(x) = \frac{1}{x^3}$ centered at $x = 2$.

key: $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!}{2^{n+3}\cdot 2\cdot n!}(x-2)^n$

$$\text{Taylor series}: f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(x)}{3!}(x-2)^3 + \cdots$$

$$f(x) = \frac{1}{x^3} \Rightarrow f(2) = \frac{1}{2^3} = \frac{2!}{2\cdot 2^3} = \frac{2!}{2^4}$$

$$f'(x) = -\frac{3}{x^4} \Rightarrow f'(2) = -\frac{3}{2^4} = -\frac{3!}{2\cdot 2^4} = -\frac{3!}{2^5}$$

$$f''(x) = \frac{3\cdot 4}{x^5} \Rightarrow f''(2) = \frac{3\cdot 4}{2^5} = \frac{4!}{2\cdot 2^5} = \frac{4!}{2^6}$$

$$f'''(x) = -\frac{3\cdot 4\cdot 5}{x^6} \Rightarrow f'''(2) = -\frac{3\cdot 4\cdot 5}{x^6} = -\frac{5!}{2\cdot 2^6} = -\frac{5!}{2^7}$$

$$\vdots$$

Then

$$f(x) = \frac{1}{x^3} = \frac{2!}{2^4} - \frac{3!}{2^5}(x-2) + \frac{\frac{4!}{2^6}}{2!}(x-2)^2 - \frac{\frac{5!}{2^7}}{3!}(x-2)^3 + \cdots$$

$$= \frac{2!}{2^4} - \frac{3!}{2^5}(x-2) + \frac{4!}{2^6\cdot 2!}(x-2)^2 - \frac{5!}{2^7\cdot 3!}(x-2)^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)!\,(x-2)^n}{2^{n+4}\cdot n!}$$

$$\text{or}\quad \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)!\,(x-2)^n}{2^{n+3}\cdot 2\cdot n!}$$

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18. (12 pts) Express $\int_0^{1/2}\cos(x^2)\,dx$ as an infinite series.

key: $\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{4n+1}}{(2n)!(4n+1)}$

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{4n}}{(2n)!}$$

Then

$$\int_0^{\frac{1}{2}}\cos(x^2)\,dx = \int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{4n}}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{4n+1}}{(2n)!\,(4n+1)}\Bigg|_0^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\left(\frac{1}{2}\right)^{4n+1}}{(2n)!\,(4n+1)}$$

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19. (8 pts) Determine whether the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}+5}{n^3-2n}$ converges or diverges. Support your answer.

key: Converges by the Limit Comparison Test.

$= a_n$

$= b_n$

$$\text{Consider the series}\quad \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{2}}} \quad \tfrac{5}{2} = p > 1$$

: converges (p-series)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}+5}{n^3-2n}}{\frac{\sqrt{n}}{n^3}} = \lim_{n\to\infty}\frac{\sqrt{n}+5}{n^3-2n}\cdot\frac{n^3}{\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{n^{\frac{7}{2}}-5n^3}{n^{\frac{7}{2}}-2n^{\frac{3}{2}}} = \lim_{n\to\infty}\frac{n^{\frac{7}{2}}}{n^{7/2}} = 1 > 0 \ : \text{converges}$$

By the Limit Comparison Test, both series must do the same thing.

And since $\sum_{n=1}^{\infty} b_n$: converges,

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}+5}{n^3-2n} : \text{converges.}$$