Notes on mathematics related to the 'buzz contest'.

Bang. It's very simple to identify numbers that are divisible by 5; their last digit is 0 or 5. So much for 'bang': any digit is 5, or the last digit is 0.

Buzz. For buzz, it's less simple. Mentally dividing by 7, fast, can be a bit of a pain. However, there are some shortcuts. Consider the case of 119. Rather than doing the long division mentally, one can throw out 70 and just consider what remains, 49. Thus, 119 is a buzz. For 168, similarly, throw out 140 and 28 remains, hence, 168 is also a buzz.

Fibby. The Fibonacci numbers are scarce. They are 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and after that, if the buzz contest ran to larger numbers (which it has not, not yet), the next one is way out at 233 and after that, at 377. There isn't any nice way to spot a Fibonacci number except to run the sequence forward and see if you hit your number. Is 920 a Fibonacci number? 144, 233, 377, 610, 987. No.

Crash. Determining whether a number is prime is a computational challenge that has only recently been satisfactorily resolved. There are good, deterministic algorithms that test for primality. If all we want is an answer that is probably correct, then there are even faster algorithms. Fermat's (little) theorem, saying that $a^{p-1} \equiv 1 \pmod{p}$ is the key here, but there are twists which improve on it. For instance, $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$. Most non-primes fail this test. For instance, to test 247 for primality, we'd take 2 to the power 123 in modular arithmetic mod 247 and see if we got plus or minus 1. This requires very little arithmetic:

$$2^1 \equiv 2, 2^2 \equiv 4, 2^8 \equiv 256 \equiv 9, 2^{16} \equiv 81 \equiv 3^4$$
$$2^{32} \equiv 3^8 \equiv 243 \cdot 27 \equiv -4 \cdot 27 \equiv -108 \equiv 139$$

and so on, and then one merely multiplies these together to obtain $2^{123} \equiv 164 \pmod{247}$. Since this is neither 1 nor 246, 247 is not prime, and we do not say 'crash'.

You are thinking this is ridiculous? Yes, it is. Well, it's ridiculous if all the arithmetic must be done in your head. It's actually a sweet piece of mathematics when compared to grinding out trial divisions, when the numbers to be tested are large. When the numbers are small, trial division is quite effective, especially when the way is smoothed with a couple of tricks.

First trick: numbers that are multiples of 3 are easy to recognize. Consider 168 again. The sum of the digits of 168 is $1 + 6 + 8 = 15$. The sum of the digits of 15 is $1 + 5 = 6$. Thus, 168 is a multiple of 3.

Second trick: numbers that are multiples of 11 are also easy to recognize. Again, consider 168. The alternating sum of the digits of 168 is $1 - 6 + 8 = 3$.

Now 3 is not 0 and it is not any other multiple of 11 either. So 168 is not a multiple of 11.

Divisibility by 2 or 5 is trivial to spot. That means that the only hard cases are to spot divisibility by 7 or 13. If those can be handled, every number out to 169 is safe, because that's the first number that is not prime, but that has only prime factors greater than 11. And 169 is 13^2, and after 169, the next number that will slip past the procedure of screening just for divisibility by primes less than 13 will be 221. Since the buzz contest has never yet reached 221, this method ought to be good enough for contest purposes.

Pop.. Product of two distinct primes. The techniques that work for 'crash' are serviceable for 'pop' too. Good luck. If you find a prime factor p of n, just think about whether n/p is prime and not p.

Whizz. Square-free. There are a lot of these. The fraction of integers that are square free happens to be, in the limit, $6/\pi^2$, a beautiful result due to Euler. But we don't need to know the general pattern. Just how to spot them. To be square free, a number must not be a multiple of 4, or 9, or 25, or 49, or 121, or —but we won't worry about the rest of it. Contest numbers don't go that high, generally speaking.

Zipp. A perfect power! These are rare, just like Fibonacci numbers. In fact, they're rarer. The higher powers of 2 are 4, 8, 16, 32, 64, 128, 256, etc. The relevant powers of 3 are 9, 27, 81, 243, etc. The powers of 5 are 125, 625, etc. Powers of 6 (beyond the 0th and 1st which don't count) are 36, 216, etc. Similarly, 49, 343, etc. Powers of 10? Why are we even talking about *that*? Beyond powers of 10, only squares will be relevant.

Squawk. Sum of two squares. By rule, (and in reality), 0 is 0^2 so it's square. Now, here's the deal: Gauss spotted a connection between sums of squares and $i = \sqrt{-1}$. When $n = a^2 + b^2$, that's because $n = (a + ib)(a - ib)$. Now if $m = c^2 + d^2$, then

$$mn = (a + ib)(a - ib)(c + id)(c - id)$$
$$= ((a + ib)(c + id))((a - ib)(c - id)) = (ac - bd)^2 + (ad - bc)^2.$$

The set of squawks is *closed under multiplication*! That means that we really only need to be able to tell squawk primes apart from non-squawk primes. Here, there's a simple rule. $2 = 1^2 + 1^2$. If p is a prime greater than 2, then p can be written as $a^2 + b^2$ if and only if p is congruent to 1 mod 4.

For contest purposes, then, one scans n to see whether a non-squawk prime divides n to an odd power. If the answer is yes, then n is not squawky. If it's no, then it is. Thus, 35 is not a squawk because 7 divides 35 to the power 1 but not 2, and 7 is congruent to 3 rather than 1 mod 4.

For further information, see

`http://en.wikipedia.org/wiki/Gaussian_integer`