

TAMU 2008 Freshman-Sophomore Math Contest
Second-year student version.

There are five problems, each worth 20% of your total score. This is not an examination, and a good score, even a winning score, can be well short of solving all five problems completely. See what you can do with these. Rules: No laptop computers, no calculators, no cell phones or other means of communicating with the outside. You're on your own for the duration of the contest. Blank paper and pencils are provided.

1. Show that

$$\int_{u=0}^{2\pi} \frac{\sin u}{u} du = \pi \int_{u=0}^{\pi} \frac{\sin u}{u(\pi+u)} du.$$

Let A be the integral on the left, and B , the integral on the right. We have

$$A = \int_0^{2\pi} \frac{\sin u}{u} du = \int_0^{\pi} \frac{\sin u}{u} du + \int_{\pi}^{2\pi} \frac{\sin u}{u} du.$$

The second integral is, by a change of variable replacing u with $u-\pi$, equal to $\int_0^{\pi} \sin(u+\pi)/(u+\pi) du$. Now $\sin(u+\pi) = \sin(u)\cos(\pi) + \sin(\pi)\cos(u) = -\sin(u)$. Thus

$$A = \int_0^{\pi} \sin u \left(\frac{1}{u} - \frac{1}{u+\pi} \right) du = \int_0^{\pi} \sin u \left(\frac{\pi+u}{u(u+\pi)} - \frac{u}{u+\pi} \right) du = B.$$

2. Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 1 - y(t) \\ \frac{dy}{dt} &= 12 - 2x(t) \end{aligned}$$

with initial conditions $x(0) = 11$, $y(0) = 8$. At some later time, ($t_1 > 0$), one of $x(t_1)$ or $y(t_1)$ hits 0, while the other is still positive. Which one remains positive? Explain.

There are several solutions. Perhaps the slickest is this: Consider $E(t) := 2(x(t) - 6)^2 - (y(t) - 1)^2$. We have

$$E'(t) = 4(x(t)-6)x'(t) - 2(y(t)-1)y'(t) = 4(x-6)(1-y) - 2(y-1)(12-2x) = 0.$$

Since $E = 0$ initially, E is always 1. The curve $E = 1$ is a hyperbola, with asymptotes being the lines corresponding to $E = 0$, that is, $y - 1 = \pm\sqrt{2}(x - 6)$. We started in the quadrant to the right of $(6, 1)$ initially, because $(11, 8)$ lies below the line $y - 1 = \sqrt{2}(x - 6)$, because $7 < 5\sqrt{2}$ because $1.4 < \sqrt{2}$. Thus, staying on the same branch of the hyperbola, we must end up at the point where this branch crosses one or the other of the axes, and it only crosses the y axis, so it is y that is zero at t_1 .

Another approach is more workmanlike, and doesn't depend on pulling rabbits from hats. Since $x' = 1 - y$, $x'' = (1 - y)' = -y' = 2x - 12$. Thus, $x(t)$ has initial conditions $x(0) = 11$, $x'(0) = -7$, and it satisfies the differential equation $x'' - 2x + 12 = 0$. This is a second-order linear differential equation with constant coefficients. The corresponding homogeneous equation is $x'' - 2x = 0$, and that has solutions $x_{1,h}(t) = e^{\sqrt{2}t}$ and $x_{2,h}(t) = e^{-\sqrt{2}t}$. Adjusting for the '12' by the method of undetermined coefficients, we see that $x(t)$ must have the form $x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 6$. Solving for c_1 and c_2 begins with plugging in the initial conditions to conclude that $c_1 + c_2 + 6 = 11$ and $\sqrt{2}(c_1 - c_2) = -7$. Adding $\sqrt{2}$ times the first equation to the second equation shows that $2\sqrt{2}c_1 = 5\sqrt{2} - 7 > 0$, and another such calculation gives a value for c_2 that is clearly positive. Thus $x(t)$ is the sum of two quantities that are positive for all t , and it must be $y(t)$ that eventually hits 0.

A third approach uses the theory of second order linear systems. There is a critical point at $(6, 1)$. It's a saddle point, and solutions flow toward the line $y - 1 = \sqrt{2}(x - 6)$ and away from the line $y - 1 = -\sqrt{2}(x - 6)$ because the eigenvalue $\sqrt{2}$ associated with the eigenvector $[1, -\sqrt{2}]$ is positive, while the eigenvalue $-\sqrt{2}$ associated with the eigenvector parallel to the upper line $y - 1 = \sqrt{2}(x - 6)$ is negative. Since we started at a point below the first line and above the second, the flow will take us out asymptotically along the second line and down and to the right. That takes $y(t)$ down toward zero, and preserves $x(t)$ to the right of 6, well clear of $x = 0$.

3. Let

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{2n^2}.$$

Here, the denominator is 2 to the power n^2 .

- (a) Show that $g(2) > 0$, $g(4) > 0$, and $g(8) > 0$. One thing to keep in mind is that an alternating series with terms that decrease in absolute value, and decrease to zero, converges, and that such a sum converges to a value between 0 and its first term.

Our sum for $g(x)$ is not like that, because the terms initially increase in absolute value. However, we can strip off any finite number of starting terms, and add those up separately, and then use this observation on the tail of the series. So, let $a(x)$ be the sum of all the terms with absolute value 1 or more, and $b(x)$, the rest.

We have $a(2) = 1$, while $b(2) = -2^{-2} + 2^{-6} - 2^{-12} \dots < 0$. Thus, $g(2) < 0$. For 4, we have $a(4) = 1 - 1 = 0$ and $b(4) = 2^{-3} - 2^{-5} + 2^{-15} \dots > 0$. For 8, we have $a(8) = 4 - 4 + 1 > 0$ and $0 < b(8) < 1$, so $g(8) > 0$.

(b) Show that $g(16) < 0$. We have

$$\begin{aligned} b(16) &= \sum_{n=5}^{\infty} (-1)^{n-1} (16^n) / 2^{n^2} \\ &= 16^5 / 2^{25} - 16^6 / 2^{36} + \dots = 2^{-5} - 2^{-12} + 2^{-21} \dots \end{aligned}$$

so that $0 < b(16) < 1/32$. As to $a(16)$, it is $2^3 - 2^4 + 2^3 - 1 = -1$. With $a(16) = -1$ and $b(16) < 1/32$, it follows that $g(16) < 0$.

(c) Show that $g(2^8) < 0$. This will be subsumed into the last part.

(d) Generalize. That is, state when $g(2^k)$ is positive, and explain. $g(2^k)$ is negative when k is a multiple of 4, and positive otherwise.

What seems to be happening is that the main stuff, $a(x)$, consists of a sum which is the same read forward and backward, apart from the business of the alternating signs. This suggests rearranging the sum so that its central term is the zeroth term, and the other terms are the $\pm l$ terms, l running from minus somewhere, to plus the same.

When $k = 4m$, we have

$$\begin{aligned} a(2^k) &= 2^{4m-1} - 2^{8m-4} + 2^{12m-9} + \dots + (-1)^{4m-1} 2^{16m^2-(4m)^2} \\ &= \sum_{j=1}^{4m} (-1)^{j-1} 2^{4mj-j^2} = \sum_{l=-2m+1}^{2m} (-1)^{l-1} 2^{4m^2-l^2}. \end{aligned}$$

The term $l = 0$ is largest, and gives a negative value because of the factor $(-1)^{l-1}$. The terms $l = \pm 1$, taken together, cancel that out. So, we are back to zero. But now the terms $l = \pm 2$ give a negative contribution, and the terms $l = \pm 3$, if the sum goes that far, give a smaller positive contribution, and so on out to the end. The terms $l = \pm 2$ give $2 \cdot 2^{4m^2-4}$ which is an even positive integer. As we observed at the outset, and alternating sum with terms that decrease in absolute value yields a total with the same sign as the first term. Thus, we get a negative integer for $a(2^{4m})$. As usual, $|b(x)| < 1$, so $g(2^{4m}) < 0$. We sketch the rest of the cases. If $k = 4m + 1$, we have

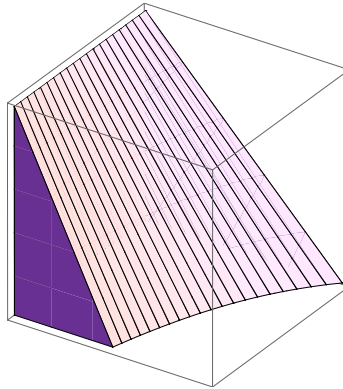
$$\begin{aligned} a(2^k) &= \sum_{j=1}^{4m+1} (-1)^j 2^{(4m+1)j-j^2} = -2^{4m^2+6m} \sum_{l=-2m+1}^{2m+1} (-1)^l 2^{l-l^2} \\ &= -2^{4m^2+6m} \sum_{l=1}^{2m+1} \left[(-1)^l 2^{l-l^2} + (-1)^{1-l} 2^{(1-l)-(1-l)^2} \right] \\ &= -2^{4m^2+6m} \sum_{l=1}^{2m+1} (-1)^l \left[2^{l-l^2} - 2^{l-l^2} \right] = 0. \end{aligned}$$

The main terms having canceled, the first term of $b(x)$ rules, and that is the $k = 4m + 2$ term, which is positive. So $g(2^{4m+1}) > 0$. For

$k = 4m + 2$, the situation is just like it is for $k = 4m$, with a central term canceled by its flanking pair of terms. But then, the largest pair of surviving near-central terms, corresponding to $k = 2m - 1$ and $k = 2m + 3$, gives a positive contribution. Thus $g(2^{4m+2}) > 0$. For $k = 4m + 3$, the terms pair off, with the middle pair being terms number $2m + 1$ and $2m + 2$, except for the first term, which remains unpaired. Thus $a = 1$, and as usual, b is too small to change the verdict. $g(2^{4m+3}) > 0$.

4. A girl walks along the edge of a 100 meter high cliff, going North at 1 meter per second, along a path 100 meters in length. Below, a ship keeps even with her, but offshore (West), a distance of $10000/(100 + t)$ meters at time t , starting at $t = 0$. Thus, initially, the line-of-sight distance from girl to ship is $\sqrt{20000}$ meters. After 100 seconds, they're both 100 meters North of where they started, but the ship is only 50 meters clear of the rocks, and the line-of-sight distance between girl and ship has diminished to $\sqrt{12500}$ meters.

What is the average value of the distance from girl to ship over the duration of her stroll?



(view from a seagull above and to the northwest of the path)

At time t , her North-South coordinates matches that of the ship. Her Up-Down coordinate differs by 100. Her East-West coordinate differs by $10000/(100+t)$, so the overall distance at time t is $\sqrt{10^4 + 10^8/(100+t)^2}$. Making first the substitution $100u = 100 + t$, and then the substitution

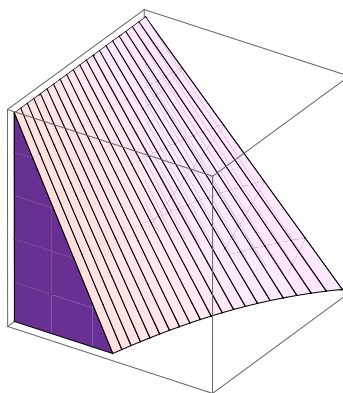
$u = \tan \theta$, the average distance A is

$$\begin{aligned} A &= \frac{1}{100} \int_{t=0}^{100} \sqrt{10^4 + \frac{10^8}{(100+t)^2}} dt = \int_{u=1}^2 \sqrt{10^4 + \frac{10^4}{u^2}} du \\ &= 100 \int_1^2 \sqrt{\frac{u^2+1}{u^2}} du = 100 \int_{\theta=\arctan 1}^{\arctan 2} \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta \\ &= \int_{\arctan(1)}^{\arctan(2)} \sec^3 \theta / \tan \theta d\theta = \int_a^b \csc \theta + \sec \theta \tan \theta \\ &= -\log(\csc \theta + \cot \theta) + \sec \theta \Big|_a^b, \end{aligned}$$

where $a = \arctan 1 = \pi/4$, and $b = \arctan(2)$. Now in view of $\cot \arctan x = 1/x$, $\sec \arctan x = \sqrt{1+x^2}$, and $\csc \arctan x = \sqrt{1+x^2}/x$, this gives an average distance of

$$\begin{aligned} &100 \left(-\log(1 + \sqrt{1+x^2}) + \log x + \sqrt{1+x^2} \Big|_1^2 \right) \\ &= 100 \left((-\log(1 + \sqrt{5}) + \log 2 + \sqrt{5}) - (-\log(1 + \sqrt{2}) + \sqrt{2}) \right). \end{aligned}$$

Further whimsical question: is the girl's name Lorelei? Why, or why not?



(view from a seagull above and to the northwest of the path)

5. The region shown is bounded by the lines from the ship to the girl as she walked the path, by lines from the shore at the base of the cliff below the girl, to the ship, by the face of the cliff itself for that 100 meter stretch, and by the triangles at either end of the region. Although the top surface of the region is composed of straight lines, it's a curved surface. Set up, but do not evaluate, a double integral for the area of the top surface.

There are a couple of approaches. One could take the base to be the square cliff face East of the lines from ship to cliff-edge. Then, the equation for

the surface would be $y = 100(100 - z)/(100 + x)$, and our integral would be

$$A = \int_{x=0}^{100} \int_{z=0}^{100} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dz dx.$$

Now

$$\frac{\partial y}{\partial x} = -\frac{100(100 - z)}{(100 + x)^2}, \quad \frac{\partial y}{\partial z} = -\frac{100}{100 + x}.$$

Thus our integral, written out in full, becomes

$$A = \int_{x=0}^{100} \int_{z=0}^{100} \sqrt{1 + \frac{10^4(100 - z)^2}{(100 + x)^4} + \frac{10^4}{(100 + x)^2}} dz dx.$$

If we view the surface as lying above the region at sea level between ship and shore, then our equation is $z = 100(1 - y(100 + x)/10^4)$. Thus, z declines linearly from 100, when $y = 0$, to 0, when $y = 10000/(100 + x)$, so that y ranges from 0 to 100 when $x = 0$, while y ranges from 0 to 50 when $x = 100$.

Now, our area integral is

$$A = \int_{x=0}^{100} \int_{y=0}^{10^4/(100+x)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dy dx.$$

And this time, $\partial z/\partial x = -y/100$, while $\partial z/\partial y = -(100 + x)/100$. This gives an answer of

$$A = \int_{x=0}^{100} \int_{y=0}^{10^4/(100+x)} \sqrt{1 + \frac{y^2}{10^4} + \frac{(100 + x)^2}{10^4}} dy dx.$$