TAMU 2009 Freshman-Sophomore Math Contest

First-year student version

There are five problems, each worth 20% of your total score. This is not an examination, and a good score, even a winning score, can be well short of solving all five probelms completely. See what you can do with these. Rules: no aids to calculation, no cell phones or other means of communicating with the outside world. You're on your own for the duration of the contest. Blank paper and pencils are provided.

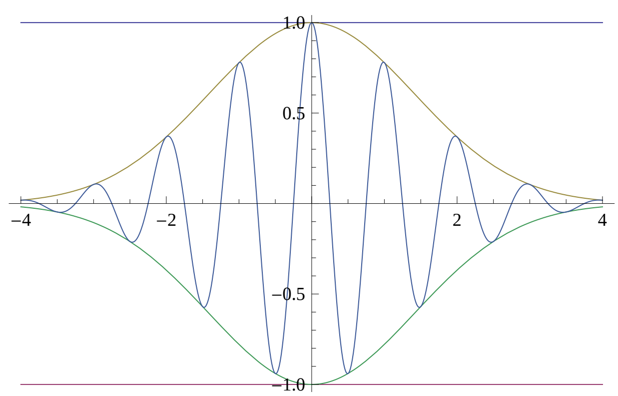
1. Consider the function $\phi(x)$ given by

$$\phi(x) = e^{-x^2/4}\cos(2\pi x).$$

(a) Find $\phi'(x)$. That would be

$$-(1/2)e^{-x^2/4}x\cos(2\pi x) - 2e^{-x^2/4}\pi\sin(2\pi x) =$$
$$-(1/2)e^{-x^2/4}\left(x\cos(2\pi x) + 4\pi\sin(2\pi x)\right).$$

(b) Sketch the graph of $y = \phi(x)$ for the interval $-4 \le x \le 4$. The graph of $e^{-x^2/4}$ is supplied; your sketch goes into the same plot window. [Here's the computer-generated sketch.]



(c) The minimum value of $\phi(x)$ occurs at an x-value somewhere between $1/4$ and $3/4$. Is this value less than $1/2$, equal to $1/2$, or greater than $1/2$? Explain. Ideally, prove your answer. The minimum (it's global) occurs somewhere that $\phi'(x) = 0$. The first factor in our expression for $\phi'(x)$ is never zero, so that place must be somewhere that the other factor is zero. Simplifying leads to this condition: $-x/(4\pi) = \tan(2\pi x)$. Since the left hand side of this equation is negative when $x > 0$, the right hand side must also be negative. That means $2\pi x$, the thing whose tangent is taken, is in the interval $(\pi/2, \pi)$, so x itself is in the interval $(1/4, 1/2)$. It's less than $1/2$.

2. Let

$$A = \int_{x=0}^{\infty} \frac{\sin x}{x}\, dx, \quad B = \int_{0}^{\infty} \frac{\sin(x^2)}{x}\, dx.$$

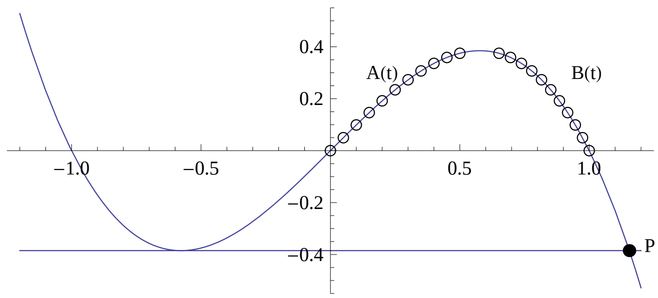
(a) Both integrals converge. Why? One answer is that the sine wave alternates, so the region between the curve and the x-axis is positive, then negative but less so, then positive but still less, and so on. This means that we have an alternating series of terms that decrease in absolute value, and such series converge. Another answer involves integration by parts. If you take $U = 1/x$, and dV to be the sine wave, then $dU = -1/x^2$, while V is either $-\cos(x)$, which is at any rate no more than 1, or something else that is bounded. The reason it's bounded is that the integral of $\sin(x^2)$ is another of those alternating series situations.

(b) Nothing in your studies up to now is likely to have equipped you to find either A or B. Nevertheless, it is possible to find A/B without any methods beyond the scope of introductory calculus. Find A/B.

In the integral defining B, make the change of variable $x^2 = u$. Equivalently, $x = u^{1/2}$, and then $dx = (1/2)u^{-1/2}$. That gives

$$\begin{aligned} B &= \int_{u=0}^{\infty} u^{-1/2} \sin(u) \cdot (1/2)u^{-1/2}\, du \\ &= \frac{1}{2} \int_{u=0}^{\infty} \frac{\sin(u)}{u}\, du = \frac{1}{2} A. \end{aligned}$$

3. Let $g(x) = x - x^3$. The graph of $g(x)$, together with other features of this problem, is shown on the interval $[-1.2, 1.2]$.



(a) Find the (exact) value of the point P at which the tangent line to the curve at its local minimum intersects the curve again.

In the spirit of calculus, let's take the derivative and set it to zero to find out where that local minimum is. We have $g'(x) = 1 - 3x^2$, and this is zero at $x = -1/\sqrt{3}$. There, $g(x) = -2/(3\sqrt{3})$. Now we have to solve $x - x^3 = -2/(3\sqrt{3})$ and find the positive value of x satisfying that equation. Let $b = -2/(3\sqrt{3})$ and $a = -1/\sqrt{3}$.

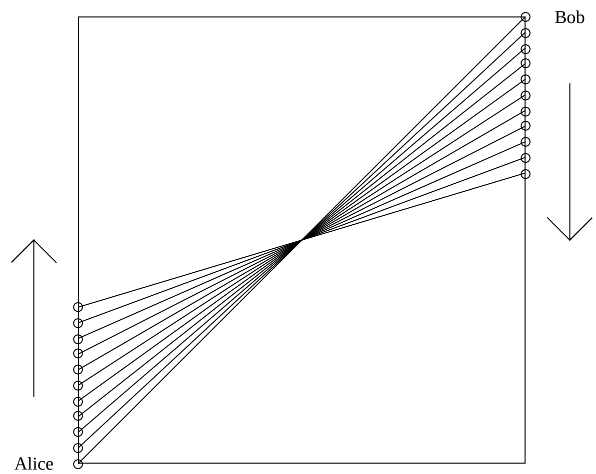
Solving cubics is no fun in general, but here we have a bit of help. Finding x so that $x - x^3 = b$ is the same as finding x so that $x^3 - x + b = 0$. We know that $x = a$ is one solution. So divide $(x - a)$ into $x^3 - x + b$ and arrive at $x^2 + ax - 2/3 = 0$. This is a quadratic and with routine calculation you get $x = a$ (again), or $x = -2a = 2/\sqrt{3}$. The required point is $(2/\sqrt{3}, -2/(3\sqrt{3}))$.

(b) Show that if u and v are distinct real numbers so that $g(u) = g(v)$ then $u^2 + uv + v^2 = 1$.

If $g(u) = g(v)$ then $u^3 - u = v^3 - v$ so $u^3 - v^3 = u - v$. Dividing by $u - v$ is OK so long as $u \neq v$, and then $u^2 + uv + v^2 = 1$.

(c) Point A starts at $(0, 0)$ at time $t = 0$ and moves up and right along the graph of $y = g(x)$. Point B starts at $(1, 0)$ and moves up and left along the graph of $y = g(x)$ so as to be at the same height as A. If the x-coordinate of A is increasing at rate 3 when $t = 0$, at what rate is the x-coordinate of B decreasing when $t = 0$? The slope on the left is 1. The slope on the right is -2. If the two points are going to keep pace in their y coordinates, then A has to move to the right twice as fast as B moves left. The x coordinate of B is decreasing at rate $3/2$.

4. Alice and Bob start at opposite corners of the campus square, which is 100 meters on a side. Alice walks north along one edge of the square at one meter per second starting at the southwest corner, and Bob walks south along the opposite edge, also at one meter per second, starting at the northeast corner.



What is their average distance from each other over the time it takes them to reach their respective new corners?

Give coordinates to the problem by saying that Alice moves from $(-50, -50)$ to $(-50, 50)$ while Bob moves from $(50, 50)$ to $(50, -50)$. The x-coordinate distance from Alice to Bob is always 100, while the y-coordinate signed distance ranges from 100 to 0 to -100. At time t, it is $2t$, if we put our $t = 0$ in the middle of the walk.

So, we need to find

$$A = \frac{1}{100} \int_{t=-50}^{50} \sqrt{100^2 + (2t)^2}\, dt.$$

With the change of variable $s = 50t$, $ds = 50\, dt$, this becomes

$$\begin{aligned} A &= \frac{1}{2} \int_{s=-1}^{1} \sqrt{100^2 + (100s)^2}\, dt \\ &= 50 \int_{s=-1}^{1} \sqrt{1 + s^2}\, ds = 100 \int_{s=0}^{1} \sqrt{1 + s^2}\, ds. \end{aligned}$$

Putting aside the constant factor of 100, we let B denote this last integral. The right trigonometric substitution for B is the one that taps into the identity $1 + \tan^2 = \sec^2$, so we put $s = \tan u$, $ds = \sec^2 u\, du$ and get $B = \int_{u=0}^{\pi/4} \sec^3 u\, du$.Integrating this by parts with $U = \sec(u)$ and $dV = \sec^2(u)\, du$ gives $B = \sec(u)\tan(u)|_0^{\pi/4} - \int_0^{\pi/4} \sec(u)\tan^2(u)\, du$. Using

$\tan^2 u = \sec^2 u - 1$ gives $2B = \sqrt{2} + \int_0^{\pi/4} \sec(u)\,du = \sqrt{2} + \ln(1+\sqrt{2})$ so $B = \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$. The answer is thus $50(\sqrt{2}+\ln(1+\sqrt{2}))$, which works out to about 114.8 meters average distance.

The integral arising here could also be found by using the identity $\sec^3 u = 1/\cos^3 u = \cos(u)/(1-\sin^2 u)^2$. Making the substitution $v = \sin u$ yields the integral of $dv/(1-v^2)$ and the methods of partial fractions work. And finally, if you know about sinh and cosh, ($\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \sinh'(x)$), you know that $\cosh^2 = 1 + \sinh^2$. So you could make the substitution $s = \sinh(v)$ and arrive at $\int \cosh^2 v\,dv$ which leads to the cleanest solution.

5. Let

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}.$$

   (a) Find the radius of convergence of the series defining f.

   When $|x| \le 1$ the $n!$ in the denominator grows faster than 2^n so by the ratio test, the series converges. If $|x| = r > 1$ then the $n+1$st term divided by the nth term is $r^{2n+1}/(n+1)$. Now this tends in the limit to infinity by L'Hôspital's rule (differentiating top and bottom separately and taking the limit of $(2n+1)r^{2n}/1$). So again by the ratio test, when $|x| > 1$ the series diverges in the most dramatic way possible; the terms go to infinity. The radius of convergence is 1.

   (b) Find two integers p and q so that

$$\left|\frac{p}{q} - \int_0^{1/2} f(x)\,dx\right| < \frac{1}{10000}.$$

   By hand?! Surely you're joking? On the other *hand*, this is a rapidly converging series. Perhaps just a few terms will give sufficient accuracy.

   The first term is 1, the second is x, the third is $x^4/2$, and the fourth is $x^9/6$. Integrating from 0 to 1/2 gives 1/2, then 1/8, then 1/320, and finally $1/(60 \cdot 2^{10})$. The next term and all the following terms are smaller than this by a factor of less than 1/2, and $2^{10} > 1000$, so all the remaining terms together amount to less than 1/30000. This leaves the arithmetic problem of adding

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{320} = \frac{160+40+1}{320} = \frac{201}{320}.$$

   So $p = 201$ and $q = 320$.