Solutions, 2010 TAMU Freshman-Sophomore Math Contest
First-year student version

1. Find
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$
This is a telescoping series. $1/((n+1)(n+2)) = 1/(n+1) - 1/(n+2)$, so $1/(n(n+1)(n+2)) = 1/n(1/(n+1) - 1/(n+2)) = 1/n - 1/(n+1) - (1/2)((1/n) - 1/(n+2))$. Now summing the first piece of this gives $1/1$, while with the second piece, the first two terms survive untelescoped so we have a contribution of $-(1/2)(1+1/2) = -3/4$. Since $1 - 3/4 = 1/4$, the answer to the question is $1/4$.

2. Lake Cony has a radius of 1000 meters and fills a conical depression 100 meters deep. Water masses 1000 kg per cubic meter, and the acceleration due to gravity is 9.81 meters/second^2. A joule is the energy needed to accelerate a mass of 1 kilogram to a speed of 1 meter per second. Find the energy (expressed in joules) needed to pump lake Cony dry.

   This is a method-of-disks integral problem. The disk that is x meters off the bottom of the lake has a radius of $10x$ meters, and an area of $100\pi x^2$. It must be raised $(100-x)$ meters. Thus we have $\int_{x=0}^{100} 100\pi x^2(100-x)\,dx$ cubic-meter-meter lifts of work to do. To lift one cubic meter of water one meter is to move 1000 kgs with a force of 9.81 newtons each, up one meter, and that takes 9810 joules. Grinding out the details gives $8.175\pi E12$ joules.

3. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$, $a_1 = 1/2$, $a_2 = -1/8$, $a_3 = 1/16$, $a_4 = -5/128$, and in general, for $n \geq 1$, $a_n = -(n-3/2)a_{n-1}/n$.

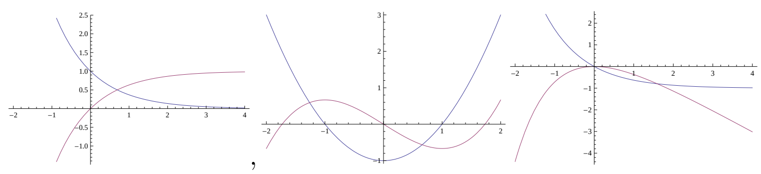
   (a) Find a_5. The rule specifying the general case gives $a_5 = -(5 - 3/2)a_4/5$, and $a_4 = -5/128$, so $a_5 = -(5-3/2)(-5/128)/5 = 7/256$.

   (b) Multiply out $f(x) \cdot f(x)$ at least to the x^4 term and then take an informed guess at a simple formula for $f(x)^2$. This would amount to expanding $(1 + x/2 - (1/8)x^2 + (1/16)x^3 - 5/128x^4 + \cdots)^2$ and this multiplies out to $1 + x + 0x^2 + 0x^3 + 0x^4 + ?x^5 + \cdots$. (The coefficient on x^4 in the expansion is $2 * (-5/128 + (1/16) * 1/2) + (-1/8)^2 = 0$, and the others are easier.) Guess: $f(x)^2 = 1 + x$.

   (c) Prove your guess. The series for $f(x)$ is the Taylor's series expansion for $(1+x)^{1/2}$ about $x = 0$ because the nth derivative at zero of $(1+x)^{1/2}$ is the product of $(1/2 - j)$ over j from 0 to $n-1$, and that's equivalent to the product of $(3/2 - k)$ over k from 1 to n, and then we have to divide by $n!$ to get the coefficient in the Taylor's series. This product obeys exactly the recursive rule given in the problem, relating a_n to a_{n-1}, because to extend the product by one step from $n-1$ to n we multiply by $n-3/2$ and then the $n!$ brings in a factor of $1/n$.

4. Suppose $g(x)$ is continuous and differentiable everywhere, and $g''(x) > 0$ for all x. Let $h(x) = \int_0^x g(t)\,dt$.

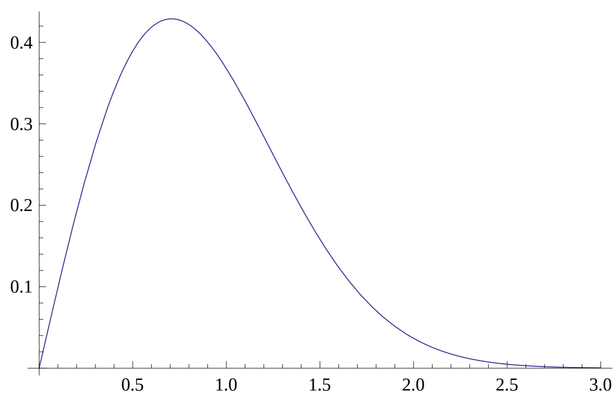
   (a) Sketch a few possibilities for the graphs of $g(x)$ and the corresponding $h(x)$.

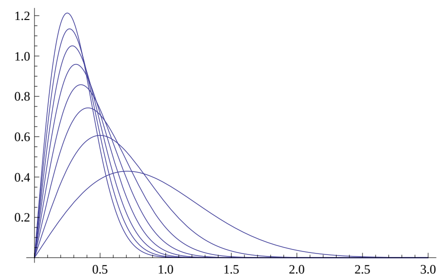
   These arise from choosing $g = e^{-x}$, $x^2 - 1$, and $e^{-x} - 1$, respectively.

   (b) Prove that the graph of $h(x)$ can cross the x axis at most three times. Between any two crossings by h of the x axis, there must be a point at which the derivative of h, which is g is zero. (Rolle's theorem). But g is concave up because its second derivative is positive, so g' is strictly increasing. Thus there can be at most one place at which $g' = 0$. Between any two zeros of g there must be one zero of g', so g can have at most two places where it's zero. That means at most three crossings of the x-axis by h, as required.

5. For $n \geq 1$, let $f_n(x) = nxe^{-nx^2}$. The graph of $f_1(x)$ is shown:



(a) Sketch the functions $f_1(x)$, $f_2(x)$, and so on, all on the same graph.



(b) For $x > 0$, find $\lim_{n\to\infty} f_n(x)$. That's zero. Using L'Hospital's rule with n as our variable, we have

$$\lim_{n\to\infty} \frac{nx}{\exp(nx^2)} = \frac{\infty}{\infty} = \frac{\lim_{n\to\infty}(d/dn)(nx)}{\lim_{n\to\infty}(d/dn)\exp(nx^2)}$$
$$= \frac{x}{\lim_{n\to\infty} x^2 \exp(nx^2)} = \frac{1}{x\infty} = 0.$$

(c) Find $\int_{x=0}^{\infty} f_n(x)\,dx$. With the change of variable $u = nx^2$, $du = 2nx\,dx$ each of these integrals becomes $\int_0^{\infty}(1/2)e^{-u}\,du = 1/2$. All the integrals evaluate to $1/2$. The moral of the story is that the integral of the limit need not be equal to the limit of the integrals.