TAMU Freshman-Sophomore Math Contest, 2011
Solutions to Both Versions

Problems on this page are common to students taking both versions of the contest.

1. Let $F(x)$ be the function given implicitly by the condition that for all real x,

$$\int_2^{F(x)} \frac{t^2+1}{t^3+1} \, dt = x.$$

Find $F'(x)$ at $x = 0$. When $x = 0$, $F(x) = 2$ because that way the integral evaluates to zero. Now differentiating both sides of the original equation and using the chain rule plus the fundamental theorem of calculus gives $F'(F^2+1)/(F^3+1) = 1$. Plug in $F = 2$ to arrive at $F' = 9/5$.

For another solution, and one that doesn't smuggle in the (correct) assumption that there is no other choice for $F(0)$ but 2, observe that if we put $G(y) = \int_2^y (t^2+1) dt/(t^3+1)$, then G is a strictly increasing, differentiable function that maps $(-1, \infty)$ onto $(-\infty, \infty)$. Solving $x = G(y)$ for y in terms of x is just taking the inverse function to G. Now $x = 0$ when $y = 2$, so $F(0) = 2$. Also, by the fundamental theorem of calculus, $dG/dy = 5/9$ at $y = 2$. So $dF/dx = 9/5$ at $x = 0$.

2. Lines are drawn from $(0, c)$ to $(1 - c, 0)$ for each c, $0 \le c \le 1$. (see figure) Find the equation of the curve to which all these lines are tangent.

One approach is to observe that if $y = f(x)$ is the equation of the curve, then $f(x)$ is the maximum value of the height at x of any of the lines in use. The equation of the line through $(0, c)$ and $(1 - c, 0)$ is $y = -cx/(1 - c) + c$. To find the c that maximizes this, take the derivative with respect to c of $-cx/(1 - c) + c$. That works out to $1 - x/(1 - c)^2$, which must therefore be zero at the optimal choice of c. (Taking $c = 0$ or $c = 1$ is clearly inferior or not relevant.) This leads to $c = 1 - \sqrt{x}$. and $y = (\sqrt{x} - 1)x/\sqrt{x} + (1 - \sqrt{x}) = x - 2\sqrt{x} + 1$.

For a second solution, reasoning from the diagram, it appears that the points on the curve are points where 'adjacent' lines intersect.

The lines joining $(0, c)$ to $(1 - c, 0)$ and $(0, c + \epsilon)$ to $(1 - c - \epsilon, 0)$ intersect at $((1 - c)(1 - c - \epsilon), c(c + \epsilon))$. As $\epsilon \to 0$, this point of intersection converges to $((1 - c)^2, c^2)$. Thus the equation of the curve is given parametrically by $x = (1 - c)^2$, $y = c^2$, $0 \le c \le 1$. Solving this to put y explicitly in terms of x gives $y = (1 - \sqrt{x})^2$ or if you like them multiplied out, $y = 1 - 2\sqrt{x} + x$. That solves the problem, but using a new system of coordinates with $x = (u + v)/2$ and $y = (-u + v)/2$ transforms this to the simpler equation $v = (1 + u^2)/2$. This shows that the curve is part of a parabola.

One can justify geometrically the observation in the first sentence of the solution above. Any point that lies beneath one of the lines cannot be on the curve. If lines A, B, C arise out of choices $c, c + \epsilon, c + \delta$ where $\delta > \epsilon$, then the intersection of lines A and C lies below B. (This is a straightforward if somewhat tedious calculation.) Thus, points that are intersections of two distinct lines cannot be on the curve.
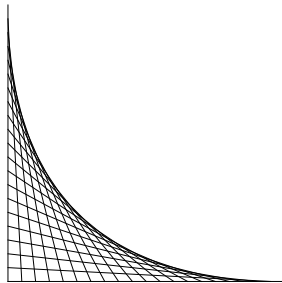
3. Suppose (a_i) is a doubly infinite sequence of nonnegative numbers with the property that for all integers i, positive, zero, or negative, $a_i^2 \geq a_{i-1}a_{i+1}$. Assume further that no zero entry comes between any two positive entries.

   (a) Prove that $a_{i-1}a_{i+2} \leq a_i a_{i+1}$ and $a_{i-2}a_{i+2} \leq a_{i-1}a_{i+1}$ for all i. Judging from the second part of the problem, there may be infinitely long stretches of zeros in (a_i). So first we think about what happens when, say, $a_{k-1} = 0$ but $a_k > 0$. It is given that no zero entry comes between any two positive entries. Thus, all a_j with $j < k$ are zero. Similarly, if $a_l > 0$ and $a_{l+1} = 0$ then $a_m = 0$ for all $m > l$. That is, the positive entries, if any, come in one stretch of consecutive values of k.

   The inequalities to be proved are safe bets if any of the a's involved are zero, because of this single-stretch property. So assume that all a's are positive. Then our premise that $a_j^2 \geq a_{j-1}a_{j+1}$ is equivalent to $a_j/a_{j-1} \geq a_{j+1}/a_j$. But then

   $$\frac{a_i}{a_{i-1}} \geq \frac{a_{i+1}}{a_i} \geq \frac{a_{i+2}}{a_{i+1}}$$

   and cross multiplying the ends of these gives $a_i a_{i+1} \geq a_{i-1}a_{i+2}$. Stretching the big chain of inequalities one step more to include also $a_{i-1}/a_{i-2} \geq a_i/a_{i+1}$ gives the other required inequality.

   (b) The number triangle shown (in part) below is constructed by setting each entry in any row below the first to be the sum of the numbers above it and immediately to the left, directly above it, and above it

and immediately to the right.

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 \\
0 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & 0 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1
\end{array}
$$

Prove that within each row, the square of any entry is greater than or equal to the product of the entries to its immediate right and left. (Thus, for instance, $16^2 \geq 19 \cdot 10$.) Let a_i's be the entries in one row, and b_i's be the entries in the next row. Then $b_j = a_{j-1} + a_j + a_{j+1}$, so that the difference $b_i^2 - b_{i-1}b_{i+1}$, call it d, is given by the expression

$$
\begin{aligned}
d = & a_{i-1}^2 + a_i^2 + a_{i+1}^2 + 2a_{i-1}a_i + 2a_{i-1}a_{i+1} \\
& + 2a_i a_{i+1} - a_{i-2}a_i - a_{i-2}a_{i+1} - a_{i-2}a_{i+2} - \\
& a_{i-1}a_i - a_{i-1}a_{i+1} - a_{i-1}a_{i+2} - \\
& a_i^2 - a_i a_{i+1} - a_i a_{i+2}.
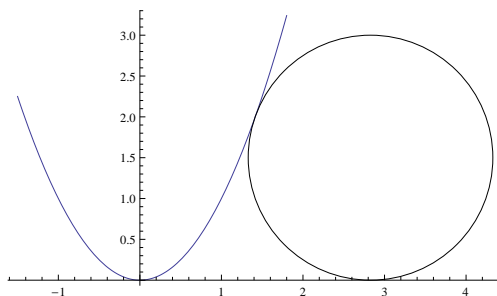\end{aligned}
$$

Canceling and simplifying yields

$$
\begin{aligned}
d = & (a_{i-1}^2 - a_{i-2}a_i) + (a_{i-1}a_i - a_{i-2}a_{i+1}) \\
& + (a_i a_{i+1} - a_{i-1}a_{i+2}) + (a_{i+1}^2 - a_i a_{i+2}) \\
& + (a_{i-1}a_{i+2} - a_{i-2}a_{i+2}).
\end{aligned}
$$

All terms here are nonnegative from the first part.

The next three problems are specific to the Freshman version of the contest.

4. Find the equation of the circle tangent to the x axis and tangent to the parabola $y = x^2$ at the point $(\sqrt{2}, 2)$.


The center is $(2\sqrt{2}, 3/2)$ and the radius is $3/2$. This follows from algebra after observing that the tangent line to both the circle and the parabola, at the given point of intersection, has slope $2\sqrt{2}$ because the derivative of x^2 at $x = \sqrt{2}$ is $2\sqrt{2}$. The line joining the point of tangency to the center of the circle thus has slope $1/(2\sqrt{2})$, and it goes to a point (a, r) where r is the radius of the circle.

5. Find
$$\int_0^1 (1-x)^{10} x^3 \, dx$$
and simplify to a single rational number with explicit numerator and denominator. The value is $1/4004$. The change of variables $u = 1-x$ greatly simplifies the computation and the intermediate answer is $1/11 - 3/12 + 3/13 - 1/14$.

6. Consider a regular octahedron with edges of length 2:
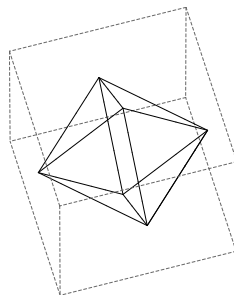
   (a) Find the volume of the octahedron. The center of the octahedron is the center of the square formed by the four vertices not part of any pair of opposite vertices. This square has area 4, and the volume of the solid is $(2/3) * \text{height} * \text{base area}$. The height is the distance from the center of the square to any of its corners, because by symmetry the distance from the center to any vertex is the same as to any other. That distance is $\sqrt{2}$, so the volume is $8\sqrt{2}/3$.

   (b) Find its surface area. The faces are equilateral triangles with side 2 and height $\sqrt{3}/2$. They thus have area $\sqrt{3}$ each, and the overall area is $8\sqrt{3}$.

The last three problems are specific to the Sophomore version of the contest:

7. Find
$$\int_0^1 \frac{1-t^2}{(1+t^2)^2} \, dt.$$
Set $t = \tan\theta$, $dt = \sec^2\theta \, d\theta$. The integral becomes
$$\int_0^{\pi/4} (1 - \tan^2\theta) \, d\theta / \sec^2\theta.$$

This simplifies to $\int_0^{\pi/4} \cos^2\theta - \sin^2\theta \, d\theta$. Using the trigonometric identity $\cos^2\theta - \sin^2\theta = \cos(2\theta)$, this last integral becomes $\int_0^{\pi/4} \cos(2\theta) \, d\theta = (1/2)$.

8. Let $f(x) = \sum_{n=0}^{\infty} x^n 10^{-n^2}$.

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(a) Determine, with proof, whether or not the series defining f converges for all x. The ratio of the $(n+1)$th term to the nth term is $10^{-(2n+1)}x$. This goes to zero as n goes to ∞, whatever the value of x. Thus by the ratio test, the series converges for all x.

(b) Find $f(2)$, accurate to within $\pm 10^{-20}$. When $|a_{n+1}| \leq r|a_n|$ for $n \geq N$, $|\sum_N^\infty a_n| \leq |a_N|/(1-r)$. Our a_n is $2^n 10^{-n^2}$. For $n \geq 1$, and $x = 2$, the ratio $10^{-(2n+1)}x$ is less than or equal to $1/500$. Thus $\sum_N^\infty a_n \leq (500/499)a_N$. Taking $N = 5$ will suffice to ensure that the 'tail' of the series comes to less than 10^{-20}. Thus we need only find $s = a_1 + a_2 + a_3 + a_4$. A little arithmetic gives

$$s = 1.2004000080000016,$$

or if we wanted more digits,

$$s = 1.20040000800000160000000032000000000640000,$$

give or take *not much*.

9. A circular wheel of radius 1 has a red dot at one point on its rim. It rolls counterclockwise, without slipping, around the outside of the fixed circle $x^2 + y^2 = 4$. Initially, the dot is touching the fixed circle and is thus at $(2,0)$. When the wheel has rolled 1/8th of the way around the fixed circle, find

   (a) the (x, y) coordinate of the red dot.
   (b) The direction in which the dot is then moving.

The dot is at $(2\sqrt{2}, 1\sqrt{2})$ and is moving straight 'up', parallel to the y axis.
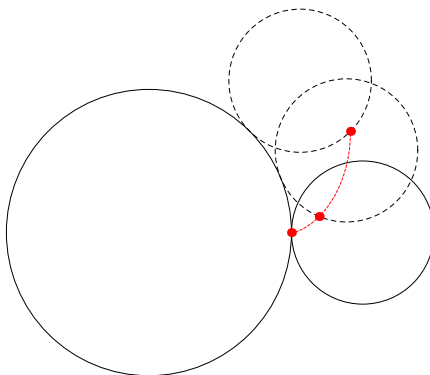
In general, when the angle between the x axis and the line joining the centers of the two circles is θ, the distance along the rim of the big circle that the little circle has rolled is 2θ. Thus, not only has it spun (counterclockwise) through an angle of θ, just by rotating about the big circle,

it has rolled another 2θ radians. This means that as a function of θ, the dot's position is $(3\cos\theta, 3\sin\theta) + (-1\cos(3\theta), -1\sin(3\theta))$. Setting $\theta = \pi/4$ gives the position, and taking the derivative gives the velocity vector. Any parallel vector can be used to describe the direction of motion.