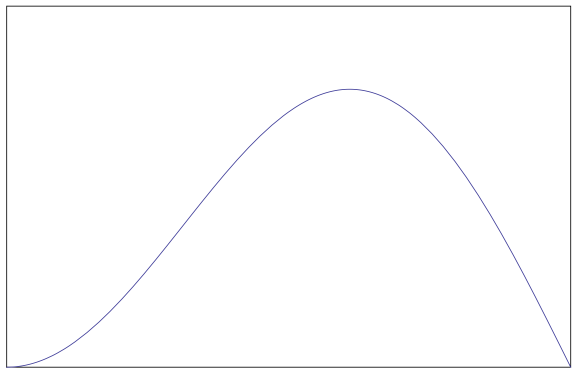
# Freshman-Sophomore Contest 2012

Solutions, First-Year Student Version

1. Let $f(x) = \sin x \sin 2x$.
   (**a**) Sketch the graph of $f(x)$ on the interval $0 \le x \le \pi/2$.



   (**b**) Find $\int_{x=0}^{\pi/2} f(x)\,dx$. Note that $\sin 2x = 2\sin x \cos x$. Thus with the substitution $u = \sin x$, $du = \cos x\,dx$, the original integral works out to $\int_{u=0}^{1} 2u^2\,du = 2/3$.
   (**c**) Find the maximum of $f(x)$ on the interval $0 \le x \le \pi/2$. The original function has derivative $2\sin x \cos^2 x - \sin^3 x$ and using $\cos^2 x = 1 - \sin^2 x$ this simplifies to $2\sin x - 3\sin^3 x$. The maximum does not occur at the ends of the interval because the function is zero there and positive in between, and $f$ is differentiable, so the maximum occurs at some point in the open interval $(0, \pi/2)$ at which $2\sin x - 3\sin^3 x = 0$. Since $\sin x \neq 0$ in this interval, we must have $2 - 3\sin^2 x = 0$. That means $\sin x = \sqrt{2/3}$, so $\cos x = \sqrt{1/3}$, so $f(x) = 2(2/3)/\sqrt{3}$. Tidied up, that's $(4/9)\sqrt{3}$, which is the answer.

2. Let $g(t) = \exp(-t^2/2)$ and let $G(x) = \int_{t=0}^{x} g(t)\,dt$.
   (**a**) Give the Taylor's series expansion of $g(t)$ about zero, that is, write $g(t)$ in the form

$$g(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

   and give a formula for the general coefficient $a_n$ in terms of $n$. The power series expansion for $e^z$ is $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$. Taking $z = -(1/2)t^2$ gives $g(t) = 1 + 0t - (1/2)t^2 + 0t^3 + (1/2!)(1/2)^2 t^4 + 0t^5 - (1/3!)(1/2)^3 t^6 + \cdots$. The coefficient on $t^{2m+1}$ is zero, and the coefficient on $t^{2m}$ is $(-1)^m 2^{-m}/m!$.
   (**b**) Likewise, give the Taylor's series expansion of $G(x)$ about zero. The series expansion of an integral is the term by term integral of the series expansion of the function that was integrated. Here, that means that

$G(x) = x - (1/6)x^3 + (1/40)x^5 - (1/336)x^7 + \cdots$; the coefficient on $x^{2m}$ is zero and the coefficient on $x^{2m+1}$ is $(-1)^m 2^{-m}/(2m+1)m!$.
(**c**) Find a decimal approximation to $G(1/2)$, accurate to within $\pm 10^{-3}$. ( Of course, $G(1/2)$ is not equal to $\pi$, but if it were, you'd answer 3.142.) Only the first three terms are needed, because after that the terms are much less than $10^{-3}$ and since the sum is alternating with terms decreasing, the absolute value of the first term left out gives an upper bound for the error. The arithmetic is that $1/2 = 0.5000$, $1/48 \approx 0.0208$, and $1/1280 \approx 0.0008$. Adding and subtracting terms gives 0.4800 which is more than sufficient for three places: the answer is 0.480. With a computer, more digits can be got easily enough along the same lines; for instance, 0.4799252190.

3. Consider the sequence $(c_n)$ given for $n \geq 2$ by $c_n = 2\ln(n) - \ln(n^2-1)$. (Thus, $c_2 = 2\ln 2 - \ln 3$.) Find, with proof, $\sum_{n=2}^{\infty} a_n$. This is a telescoping sum. First, observe that $\ln(n^2-1) = \ln((n-1)(n+1)) = \ln(n-1) + \ln(n+1)$. Next, look at the first several terms from this perspective:

$$\begin{aligned}&2\ln 2 - \ln 1 - \ln 3 + 2\ln 3 - \ln 2 - \ln 4 + 2\ln 4 - \ln 3 - \ln 5\\ &+ 2\ln 5 - \ln 4 - \ln 6 + 2\ln 6 - \ln 5 - \ln 7 + \cdots ;\end{aligned}$$

Apart from one of the original $\ln 2$'s, everything cancels. The value is $\ln 2$, or numerically, about 0.69. A more rigorous solution would go into why the sum converges. If one quits at the term $2\ln n - \ln(n-1) - \ln(n+1)$, then the partial sum, ending with that term, has mostly canceled, with $\ln 2 + \ln n - \ln(n+1)$ left over. Since $\ln n - \ln(n+1) = \ln(n/(n+1))$ and $n/(n+1)$ tends to 1 as $n$ tends to infinity, and since $\ln 1 = 0$ and the log function is continuous, this means that $\ln n - \ln(n+1)$ goes to zero so the partial sums converge to $\ln 2$ as $n$ tends to infinity.

4. Find
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+3)}.$$

The tool here is partial fractions, which converts the problem to a telescoping sum. The partial fraction decomposition of $1/(n(n+1)(n+3))$ has the form $A/n + B/(n+1) + C/(n+3)$. Setting $n = 0$ and evaluating $1/(n+1)(n+3)$ gives $A = 1/3$. Similarly, $B = -1/2$ and $C = 1/6$. Thus we can write our sum as

$$\lim_{N\to\infty} \frac{1}{6} \sum_{k=1}^{N} \left( \frac{2}{k} - \frac{3}{k+1} + \frac{1}{k+3} \right).$$

In general the fraction $1/n$ appears in this sum twice as $/1k$ with $k = n$, three times, with a negative sign on it, as $1/n = 1/(k+1)$, and once with a positive sign, as $1/n = 1/(k+3)$. Everything cancels. But at the outset, $1/1$ does not appear as $1/(k+1)$ with $k = 0$, nor does it appear as $1/(k+3)$ with $k = -2$. We get 2 copies of $1/1$, uncanceled. Likewise, $1/2$ appears twice with a plus sign, as $1/k$ with $k = 2$, and three times with a minus sign, as $1/(k+1)$ with $k = 1$. But not as $1/(k+3)$. Similarly, $1/3$ appears twice with a plus and three times with a minus, so it contributes $-1/3$. So the contribution of 1 and $1/2$ $1/3$ is $2 - 1/2 - 1/3 = 7/6$.

At the other end, there are also some loose ends. But these tend to zero as $N$ tends to infinity. The sum is $7/6$. That makes the answer $7/36$.

Another, less formal way to look at all this is to just write out several terms and inspect for cancellation. Thus, putting aside the factor of $1/6$ that sits in front of it all, we have

$$\begin{aligned} &\left(\frac{2}{1} - \frac{3}{2} + \frac{1}{4}\right) + \left(\frac{2}{2} - \frac{3}{3} + \frac{1}{5}\right) \\ &+ \left(\frac{2}{3} - \frac{3}{4} + \frac{1}{6}\right) + \left(\frac{2}{4} - \frac{3}{5} + \frac{1}{7}\right) + \left(\frac{2}{5} - \frac{3}{6} + \frac{1}{8}\right) + \cdots . \end{aligned}$$

Look at the fractions with denominator $1/4$ or $1/5$: there are three of them, with numerators 2, then $-3$, and finally, $+1$. Look at the fractions with denominator 1, 2, and 3. They're different; they don't cancel out like that. And so in the limit the sum is $7/6$, and putting back our factor of $1/6$ that we set aside, the answer is $7/36$.

5. A vein of ore in a gold mine tapers down to a point at a depth of 2000 meters; it's a cone with a surface radius of 100 meters. Assuming the rock at depth $h$ meters yields $h/200$ troy ounces of gold per cubic meter, how much gold will the vein yield by the time it's exhausted? (In troy ounces.) For kicks, about how much money would that be? An integral is in order: let $t = 2000 - h$; $t$ represents how far you are from the bottom of the

hole. The cross sectional radius at height $t$ is $t/20$ so that when $t = 2000$, $r = 100$. The yield is $h/200 = (2000 - t)/200$. So we compute

$$\int_{t=0}^{2000} \frac{2000 - t}{200} \pi \left(\frac{t}{20}\right)^2 dt = \frac{50000000}{3}\pi.$$

At \$1600 per troy ounce, that's about 80 billion dollars. Then again, digging that hole is going to be expensive.

6. An ant crawls around a circle of radius 10 cm, starting at some point $A_0$. It crawls at a uniform speed until it gets back to where it started. Find the average value of the square of the straight-line distance (measured in centimeters) across the disk from $A_0$ to the ant's position as it crawls.

   Parametrize the path of the ant as $10(\cos t, \sin t)$, as $t$ runs from 0 to $2\pi$. The square distance to $(10, 0)$ is then $100((1 - \cos t)^2 + \sin^2 t)$. The required average is thus $\frac{100}{2\pi} \int_{t=0}^{2\pi} (1 - \cos t)^2 + \sin^2 t \, dt$. The $\cos^2 + \sin^2$ stuff simplifies to 1, the integral of $\cos t$ cancels to 0, and the upshot is that the answer is 200.