Solutions of 2018 Power Team
Texas A&M High School Mathematics Contest
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We will write the base ten expansion of a number $n$ by $\overline{a_k a_{k-1} \dots a_1 a_0}$, so that $0 \le a_i \le 9$ and $n = \sum_{i=0}^{k} a_i 10^i$.

Let us prove the following simple properties of $S(n)$, which will be used later.

**Lemma 1.** *The difference $S(n) - n$ is divisible by 9.*

*Proof.* If $n = \overline{a_k a_{k-1} \dots a_1 a_0}$, then $n - S(n) = 10^k a_k + 10^{k-1} a_{k-1} + \dots + 10 a_1 + a_0 - a_k - a_{k-1} - \dots - a_1 - a_0 = (10^k - 1) a_k + (10^{k-1} - 1) a_{k-1} + \dots + 9a_1$. But $10^i - 1 = 99\dots9$ is obviously divisible by 9. $\square$

**Lemma 2.** *For every positive integer $n$ we have $S(n) \le 9(\lfloor \log n \rfloor + 1)$, where $\lfloor x \rfloor$ denotes the largest integer not greater than $x$, and $\log x$ is the decimal logarithm.*

*Proof.* If $n = \overline{a_k a_{k-1} \dots a_1}$ has $k$ digits, then $10^{k-1} \le n < 10^k$, so $k - 1 \le \log n < k$, i.e., $\lfloor \log n \rfloor = k - 1$. We also have $S(n) = a_k + a_{k-1} + \dots + a_1 \le 9k = 9(\lfloor \log n \rfloor + 1)$. $\square$

**Problem 1.**

By Lemma 2, we have $S(5^n) \le 9(\lfloor \log 5^n \rfloor + 1) = 9(\lfloor n \log 5 \rfloor + 1) < 9(0.7n + 1)$. So, we have $2^n \le 6.3n + 9$, which implies $n \le 5$. It follows that we have to check $n = 1, 2, 3, 4, 5$. The corresponding values of $5^n$ are $5, 25, 125, 625, 3125$. Their sums of digits are $5, 7, 8, 13, 11$. So, the answer is $n = 3$.

**Problem 2.**

Using Lemma 2, we get $S(2018^{2018}) \le 9(\lfloor 2018 \cdot \log 2018 \rfloor + 1) = 9 \cdot 6670 = 60030$. If $n \le 60030$, then $S(n) \le S(59999) = 41$. Consequently, $S(S(2018^{2018})) \le 41$, therefore $S(S(S(2018^{2018}))) \le S(39) = 12$.

The residue of division of 2018 by 9 is 2. The residues of $2018^m$ modulo 9 are therefore the same as for $2^m$, which are (for $m = 0, 1, 2, \dots$): $1, 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5, \dots$. This sequence is periodic with period 6. Since the residue of 2018 modulo 6 is 2, we conclude that $2018^{2018}$ gives residue 4 modulo 9. Among the numbers $1, 2, \dots, 12$ only 4 will have this residue. Consequently, the answer is 4.

**Problem 3.**

We have $n \le 2017$. It follows that $S(n) \le S(1999) = 28$. Hence, $S(S(n)) \le 10$, and $S(S(S(n))) \le 9$. Consequently,

$$n = 2018 - S(n) - S(S(n)) - S(S(S(n))) \ge 2018 - 28 - 10 - 9 = 1971.$$

If $r$ is the residue of $n$ modulo 9, then the residue of $n + S(n) + S(S(n)) + S(S(S(n)))$ is $4r$. Note that $4r$ is congruent to 2 modulo 9 if and only if $r$ is congruent to 5 modulo 9. The residue of 2018 is 2, which implies that $r = 5$. It follows that the only candidates for $n$ are

$$1976, 1985, 1994, 2003, 2012.$$

The corresponding values of $n + S(n) + S(S(n)) + S(S(S(n)))$ are

$$\begin{aligned}
1976 + 23 + 5 + 5 &= 2009\\
1985 + 23 + 5 + 5 &= 2018\\
1994 + 23 + 5 + 5 &= 2027\\
2003 + 5 + 5 + 5 &= 2018\\
2012 + 5 + 5 + 5 &= 2027.
\end{aligned}$$

The answer is $n = 1985$ and $2003$.

**Problem 4.**

a) Let us prove the inequality by induction. It is obviously true if one of the numbers $m, n$ is equal to zero. Let $m, n$ be positive integers. Let $m_0$ and $n_0$ be the last digits of $m, n$, respectively. Then $m = 10m_1 + m_0$ and $n = 10n_1 + n_0$, where $m_1$ and $n_1$ are obtained from $m, n$ by erasing the last digits. We have $m_1 < m$ and $n_1 < n$, and we may assume, by the inductive hypothesis, that $S(m_1 + n_1) \le S(m_1) + S(n_1)$. If $m_0 + n_0 < 10$, then the last digit of $m + n$ is $m_0 + n_0$, and we have

$$\begin{aligned}
S(m+n) = S(10(m_1+n_1) + m_0 + n_0) &= \\
S(m_1+n_1) + m_0 + n_0 &\le \\
S(m_1) + S(n_1) + m_0 + n_0 &= S(m) + S(n).
\end{aligned}$$

If $m_0 + n_0 \ge 10$, then the last digit of $m + n$ is $m_0 + n_0 - 10$ and

$$\begin{aligned}
S(m+n) = S(10(m_1+n_1+1) + m_0 + n_0 - 10) &= \\
S(m_1+n_1+1) + m_0 + n_0 - 10 &\le \\
S(m_1) + S(n_1) + 1 + m_0 + n_0 - 10 &= \\
S(m) + S(n) - 9 &< S(m) + S(n).
\end{aligned}$$

b) It follows directly from inequality (a) that

$$S(mn) = S(\underbrace{n + \cdots + n}_{m \text{ times}}) \le mS(n).$$

Write $m = \overline{a_k a_{k-1} \ldots a_1 a_0}$. Then, using (a) and the inequality $S(mn) \le mS(n)$, we get

$$\begin{aligned}
S(mn) = S(10^k a_k n + 10^{k-1} a_{k-1} n + \cdots + 10 a_1 n + a_0 n) &\le \\
S(10^k a_k n) + S(10^{k-1} a_{k-1} n) + \cdots + S(10 a_1 n) + S(a_0 n) &= \\
S(a_k n) + S(a_{k-1} n) + \cdots + S(a_1 n) + S(a_0 n) &\le \\
a_k S(n) + a_{k-1} S(n) + \cdots + a_0 S(n) &= \\
(a_k + a_{k-1} + \cdots + a_0) S(n) &= S(m)S(n).
\end{aligned}$$

**Problem 5.**

Use Problem 4:

$$S(n) = S(1000 \cdot n) = S(125 \cdot 8n) \leq S(125)S(8n) = 8S(8n)$$

and

$$S(n) = S(10^5 n) = S(2^5 \cdot 5^5 n) \leq S(32)S(5^5 n) = 5S(5^5 n).$$

**Problem 6.**

Let us assume at first that the last digit of $x$ is not 0. We have $x(10^n - 1) = 10^n x - x$. Write $x = \overline{a_n a_{n-1} \ldots a_1}$, where we allow leading digits (e.g., $a_n$) to be equal to 0. Then

$$x(10^n - 1) = 10^n x - x =$$
$$\overline{a_n a_{n-1} \ldots a_1 \underbrace{00 \ldots 0}_{n \text{ times}}} - \overline{a_n a_{n-1} \ldots a_1} =$$
$$\overline{a_n a_{n-1} \ldots a_2 (a_1 - 1)(9 - a_n)(9 - a_{n-1}) \ldots (9 - a_2)(10 - a_1)},$$

and $S(x(10^n - 1)) = a_n + a_{n-1} + \cdots + a_2 + a_1 - 1 + 9 - a_n + 9 - a_{n-1} + \cdots + 9 - a_2 + 10 - a_1 = 9n$.

Suppose that $x$ ends with $k$ zeros. Then $x = 10^k y$, where the last digit of $y$ is not zero. We have $1 \leq y < 10^n$, hence, by the proven above, we have $S(y(10^n - 1)) = 9n$. But $S(x(10^n - 1)) = S(y(10^n - 1))$.

**Problem 7.**

We have $9 \cdot 99 \cdot 9999 \cdot \ldots \cdot \underbrace{99 \ldots 99}_{2^n} = (10-1)(10^2-1)(10^4-1) \cdots (10^{2^n} - 1)$. Let $x = (10 - 1)(10^2 - 1) \cdots (10^{2^{n-1}} - 1)$. We have then

$$x < 10 \cdot 10^2 \cdot 10^4 \cdots 10^{2^{n-1}} = 10^{1+2+2^2+\cdots+2^{n-1}} = 10^{2^n - 1} < 10^{2^n}.$$

It follows from the previous problem that the answer is $9 \cdot 2^n$.

**Problem 8.**

Denote $f(x) = x + S(x)$. If $x$ does not end with 9, then $f(x + 1) = f(x) + 2$. If $x$ ends with exactly $k$ nines, then all of them will become zeros in $x + 1$, and the last non-nine digit will increase by 1. It follows that in this case $f(x + 1) = f(x) - 9k + 2$.

Let $x$ be the largest positive integer such that $f(x) \leq n$ (it exists, since we always have $f(x) > x$). Then $f(x + 1) = f(x) + 2$, since otherwise $f(x + 1) \leq f(x) - 7 < n$, which contradicts the choice of $x$. We have either $f(x) = n$, or $f(x) \leq n - 1$. If $f(x) \leq n - 2$, then we get $f(x + 1) = f(x) + 2 \leq n$, which is a contradiction. Consequently, either $f(x) = n$ or $f(x + 1) = n + 1$.

**Problem 9.** We will prove that there exist arbitrarily large sets of numbers with the same value of $x + S(x)$. We will construct such sets by showing inductively how to construct sets $A_n$ consisting of $2^n$ numbers with the same value of $x + S(x)$.

Let $f(x) = x + S(x)$, as in the previous problem. Suppose that $10^N > n$ and consider

$$f(9 \cdot 10^N + n) = 9 \cdot 10^N + n + 9 + S(n) = 9 \cdot 10^N + 9 + f(n),$$

and

$$f(9 \cdot 10^N - n) = 9 \cdot 10^N - n + 8 + 9N + 1 - S(n) = 9 \cdot 10^N + 9 + 9N - f(n).$$

The obtained two expressions are equal if and only if $f(n) = 9N - f(n)$, i.e., if $N = \frac{2f(n)}{9}$. We can find such an $N$ if $f(n)$ is divisible by 9. Note that then $f(9 \cdot 10^N + n) = 9 \cdot 10^N + 9 + f(n)$ and $f(9 \cdot 10^N - n) = 9 \cdot 10^N + 9 + 9N - f(n)$ are also divisible by 9.

We can use now this observation to construct the sets $A_n$ inductively. Each set $A_n$ will have $2^n$ numbers with the same value of $f(x)$ such that $f(x)$ is divisible by 9. Let us start with $A_0 = \{9\}$. If $A_n$ is constructed, and $f(x) = a$ for all $x \in A_n$, where $a$ is divisible by 9, then the set $A_{n+1}$ consists of numbers $9 \cdot 10^{\frac{2a}{9}} + x$ and $9 \cdot 10^{\frac{2a}{9}} - x$ for all $x \in A_n$. Then it follows from the previous paragraph that for all elements $y \in A_{n+1}$ the value of $f(y)$ is the same and divisible by 9.

Let us show that the size of $A_{n+1}$ is be twice the size of $A_n$. It is enough to show that on each step we have that $10^{\frac{2a}{9}}$ is bigger than any element of $A_n$. But $a = f(x) > x$ for every $x \in A_n$, and $10^{\frac{2a}{9}} > a$ for every $a \geq 1$.

**Problem 10.**

Let $0 \leq k_1 < k_2 < \ldots < k_{1000}$, and consider the number $n = 10^{k_{1000}} + 10^{k_{999}} + \cdots + 10^{k_1}$. We have $S(n) = 1000$. We have $n^2 = 10^{2k_{1000}} + 10^{2k_{999}} + \cdots + 10^{2k_1} + 2\sum_{1 \leq i_1 < i_2 \leq 1000} 10^{k_{i_1} + k_{i_2}}$. If all the numbers $k_{i_1} + k_{i_2}$ for $1 \leq i_1 \leq i_2 \leq 1000$ are pairwise different, then $S(n^2) = 1000 + 2 \cdot \frac{1000 \cdot 999}{2} = 1000^2$. The numbers $k_{i_1} + k_{i_2}$ are pairwise different if the sequence $k_i$ grows sufficiently fast. For instance, we can take $k_i = 2^i$.

**Problem 11.** Squares of integers give only residues 0, 1, 4, and 7 modulo 9. Consequently, their sums of digits may give only these residues. In particular, 2018 can not be equal to $S(n^2)$, as it is congruent to 2 modulo 9. On the other hand, we have

$$S((10^k - 1)^2) = S(10^{2k} - 2 \cdot 10^k + 1) = S(\underbrace{99\ldots98}_{k}\underbrace{00\ldots01}_{k}) = 9k,$$

$$S((10^k - 2)^2) = S(10^{2k} - 4 \cdot 10^k + 4) = S(\underbrace{99\ldots96}_{k}\underbrace{00\ldots04}_{k}) = 9k + 1,$$

$$S((10^k - 3)^2) = S(10^{2k} - 6 \cdot 10^k + 9) = S(\underbrace{99\ldots94}_{k}\underbrace{00\ldots09}_{k}) = 9k + 4,$$

$$S((10^{k+1} - 5)^2) = S(10^{2k+2} - 10^{k+2} + 25) = S(\underbrace{99\ldots9}_{k}\underbrace{00\ldots025}_{k+2}) = 9k + 7,$$

which implies that all positive integers congruent to 0, 1, 4, or 7 modulo 9 are sums of digits of squares. In particular, there exists $n$ such that $S(n^2) = 2017$.