Algebra Qualifying Examination
August 12, 2025

Instructions:

- Read all problems first; make sure you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit awarded will be based on the correctness of answers and the clarity and main steps of reasoning. Answers must be legible and written in a structured and understandable manner. Do scratch work on a separate page.
- Start each problem on a new page, clearly marking the problem number on that page.
- Rings always have an identity 1 and all modules are left modules.
- Throughout, the integers are denoted $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

1. Let G be a (not necessarily finite) group of order > 2. Prove that there exists a non-identical automorphism $G \longrightarrow G$.
2. (a) Prove that a group G of order $231 = 3 \times 7 \times 11$ is a product (not necessarily direct) of a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and a normal subgroup isomorphic to $\mathbb{Z}/77\mathbb{Z}$.

   (b) Prove that there are exactly two such groups G up to isomorphism.
3. Prove that the group of transformations of $\mathbb{R}$ generated by $f_1\colon x \mapsto x + 1$ and $f_2\colon x \mapsto 2x$ is solvable.
4. Compute the following $\mathbb{Z}$-modules (decompose them into a product of cyclic modules): $\mathbb{Z}_{15} \otimes \mathbb{Z}_{14}$, $\mathbb{Z}_6 \otimes \mathbb{Z}_9$.
5. Let R be a commutative ring with 1. An R-module M is called *flat* if whenever $f\colon N \longrightarrow P$ is a injective R-module homomorphism then the induced map
   $$M \otimes_R N \longrightarrow M \otimes_R P$$
   is also injective.

   If R is a PID and M is a finitely generated R-module then show that M is flat if and only if it is torsion free.
6. Let F be a finite field with 7 elements, and let β be a root of $x^3 - 2 \in F[x]$. (a) Compute the minimal polynomial for $\alpha = \beta + 1$ over F. (b) Determine, with proof, whether or not $F(\alpha)$ is Galois over F.

7. Let $\Bbbk$ be a finite field. Prove that for every degree $d > 0$ there exists an irreducible polynomial over $\Bbbk$ of degree d.

8. Let F be a field and $f(x) = x^6 + 3 \in F[x]$. Let K be a splitting field of $f(x)$. Determine $\mathrm{Gal}(K/F)$ for $F = \mathbb{Q}, \mathbb{F}_5$, and $\mathbb{F}_7$.