Algebra Qualifying Examination
January 14, 2016

Instructions: • There are nine problems worth a total of 100 points. Individual point values are listed by each problem.

• Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner.

Notation: Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

1. (12) Prove that every group of order 255 is cyclic.

2. (12) If H is a finite normal subgroup of a group G, then the index of its centralizer $C_G(H)$ is finite.

3. (12)
   (a) Show that any subgroup of finite index in a finitely generated group is itself finitely generated.
   (b) A group is said to be locally finite if every finitely generated subgroup of the group is finite. Suppose that G is a group containing a normal subgroup K such that K and G/K are both locally finite. Show that G is locally finite.

4. (12)
   (a) Let A be an $n \times n$ matrix over $\mathbb{C}$. Prove that if $Tr(A^i) = 0$ for all $i > 0$ then A is nilpotent.
   (b) Let A and B be $n \times n$ matrices over $\mathbb{C}$. Prove that if A commutes with $AB - BA$ then $(AB - BA)$ is nilpotent.

5. (8)
   (a) Is $\mathbb{Z}[x]$ a UFD? Is it a PID? Is it a Euclidean domain?
   (b) The same questions for the ring $\mathbb{Z}[x, y]$.
   Justify your answers.

6. (10) Let A be a finitely generated abelian group.
   (a) If A is finite, prove that $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
   (b) If A is infinite, prove that, for some positive integer r, $A \otimes \mathbb{Q}$ and $\mathbb{Q}^r$ are isomorphic as $\mathbb{Z}$-modules.

7. (10) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear map defined by $T(x, y) = (x - y, y - x)$ for all $x, y \in \mathbb{R}$. Consider $\mathbb{R}^2$ to be an $\mathbb{R}[x]$-module by letting $p(x) \cdot v = p(T)(v)$ for all $p(x) \in \mathbb{R}[x]$, $v \in \mathbb{R}^2$.
   (a) Is $\mathbb{R}^2$ a cyclic $\mathbb{R}[x]$-module? (That is, is $\mathbb{R}^2$ generated by a single element as an $\mathbb{R}[x]$-module?)
   (b) Find all the $\mathbb{R}[x]$-submodules of $\mathbb{R}^2$.

8. (12) Let $\alpha = \sqrt{2 + \sqrt{2}}$ in $\mathbb{R}$.
   (a) Find the minimal polynomial f of α over $\mathbb{Q}$.
   (b) What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?
   (c) Show that $\mathbb{Q}(\alpha)$ is the splitting field of f over $\mathbb{Q}$.
   (d) Show that $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

9. (12) Let $f(x) \in \mathbb{Q}[x]$, and let G be the Galois group of f.
   (a) Suppose $f(x)$ is a polynomial of degree 2. Find all possible Galois groups G and state conditions on the coefficients of f under which each such group occurs.
   (b) Suppose $f(x)$ is a polynomial of degree 3. Prove that if G is a cyclic group of order 3, then $f(x)$ splits completely over $\mathbb{R}$.