Algebra Qualifying Exam
January 2019

Instructions:

- Read all 9 problems first; make sure that you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.

- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. Do scratch work on a separate page.

- Start each problem on a new page, clearly marking the problem number and your name on that page.

- Rings always have an identity and all R-modules are left modules.

(1) Let T be a linear operator on a nonzero finite dimensional vector space V over a field F. Assume that the only T-invariant subspaces of V are the zero subspace and V itself. Prove that the characteristic polynomial of T is irreducible over F.
(By definition, a subspace W of V is called T-invariant if $T(W) \subseteq W$.)

(2) Let G be a finite group acting on a finite set X. Assume that each point in X is fixed by at least one nonidentity element of G, and that each nonidentity element of G fixes at most two points of X. Prove that the action has at most three orbits.

(3) Let n be a positive integer and let $G = D_{2^{n+1}} := \langle r, s \mid r^{2^n} = s^2 = 1,\ sr = r^{-1}s \rangle$.
  (a) Find the ascending central series $(C_n(G))_{n\geq 0}$ of G. Explain your answer.
  (Recall that, by definition, $C_0(G)$ is the trivial group and $C_{n+1}(G)$ is the inverse image of the center of $G/C_n(G)$ under the quotient map $G \to G/C_n(G)$.)
  (b) Is G nilpotent? Justify your answer.
  (c) Is G solvable? Justify your answer.

(4) Let R be a commutative ring with $1 \neq 0$, and let S be a nonempty subset of R such that $0 \notin S$ and $ab \in S$ whenever $a, b \in S$.
  (a) Prove that there exists an ideal J of R that is maximal with respect to having empty intersection with S.
  (b) Prove that J is a prime ideal.

(5) Let R be a commutative ring with $1 \neq 0$, let M be an R-module, and let I be an ideal of R. Prove that $(R/I) \otimes_R M$ and M/IM are isomorphic as R-modules.
(IM denotes the R-submodule of M consisting of all finite sums of elements of the form im where $i \in I$ and $m \in M$.)

(6) Let R be a commutative ring with $1 \neq 0$.

(a) Suppose that we have the following commutative diagram of R-modules:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Assume that the top row is exact, and that f' is injective. Prove that the sequence

$$\ker(\alpha) \xrightarrow{f|_{\ker(\alpha)}} \ker(\beta) \xrightarrow{g|_{\ker(\beta)}} \ker(\gamma)$$

is exact.

(b) Suppose that we have the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

Assume that both rows are exact. Which of the following statements are true, and which ones are false? (You do not need to justify your response.)

(i) If α_1 is surjective, and both α_2 and α_4 are injective, then α_3 is injective.
(ii) If α_1 is surjective, and both α_2 and α_4 are injective, then α_3 is surjective.
(iii) If α_5 is injective, and both α_2 and α_4 are surjective, then α_3 is injective.
(iv) If α_5 is injective, and both α_2 and α_4 are surjective, then α_3 is surjective.

(7) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 3, and G its Galois group. Prove that if G is the cyclic group of order 3, then $f(x)$ splits completely over $\mathbb{R}$.

(8) Let p be a prime number and let F_p denote the finite field of order p. Let $f(x) \in F_p[x]$ be the polynomial $f(x) := x^p - x + 1$, and let K be the splitting field of $f(x)$ over F_p. Let $\alpha \in K$ be any root of f.

(a) Let $\beta \in K$ be another root of f. Prove that $\alpha - \beta \in F_p$.
(b) Prove that $K = F_p(\alpha)$.

(9) Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let P denote the set of all odd prime numbers, and for $p \in P$ let r_p denote p-th root of 7 in $\mathbb{R}$. Given a subset A of P, show that there exists an automorphism σ of $\overline{\mathbb{Q}}$ such that $\sigma(r_p) = r_p$ for all $p \in A$, and $\sigma(r_p) \neq r_p$ for all $p \in P - A$.