# Applied/Numerical Analysis Qualifying Exam

August 13, 2011

**Cover Sheet – Applied Analysis Part**

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name________________________________________________

**Combined Applied Analysis/Numerical Analysis Qualifier**
**Applied Analysis Part**
**August, 2011**

**Instructions:** *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

**Problem 1.** Let $I(\lambda) := \int_0^\infty e^{-\lambda t}(1+t)^\lambda dt$.
(a) State Watson's lemma.
(b) Find an asymptotic estimate for $I(\lambda)$ as $\lambda \to \infty$.

**Problem 2.** Let $L[u] = -\frac{d^2u}{dx^2} + u$, $0 \le x \le 1$. Take

$$\mathcal{D}(L) := \{u \in L^2[0,1] \,|\, u'' \in L^2[0,1],\ u'(0) = u(0),\ u'(1) = -u(1)\}.$$

to be the domain of $L$.
(a) Show that $L$ is self adjoint on $\mathcal{D}(L)$.
(b) Find the Green's function for the problem $L[u] = f$, $u \in \mathcal{D}(L)$.
(c) Show that minimizing the functional

$$D[u] = u(0)^2 + u(1)^2 + \int_0^1 (u'^{\,2} + u^2)dx,$$

over all $u$ subject to the constraint $H[u] = \int_0^1 u^2(x)dx = 1$, leads to the eigenvalue problem $L[u] = \lambda u$, $u \in \mathcal{D}(L)$. (Note: the minimization problem does not assume that $u$ is in $\mathcal{D}$.)
(d) Suppose that the boundary condition at $x = 0$ is changed to $u(0) = 0$ instead of $u(0) = u'(0)$. Is the lowest eigenvalue in the new problem larger or smaller than for the old one? Explain your reasoning.

**Problem 3.** Let $\mathcal{H}$ be a Hilbert space, with the inner product and norm being $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. If $K$ is a compact, self-adjoint operator having spectrum $\sigma(K)$, then show that, for $\lambda \notin \sigma(K)$, one has $\|(K - \lambda I)^{-1}\|_{op} = (\mathrm{dist}(\lambda, \sigma(K))^{-1}$. (Hint: use the spectral theorem for compact self-adjoint operators.)

**Problem 4.** In the following, use the Fourier transform conventions

$$\hat{f}(\omega) := \mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}d\omega.$$

State and prove the Heisenberg uncertainty principle, given that

$$\int_{-\infty}^{\infty} x|f(x)|^2dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \omega|\hat{f}(\omega)|^2d\omega = 0.$$

Be sure to state any assumptions on the smoothness and decay of $f$. Is there an $f$ that minimizes the "uncertainty product?" If so, what is it? (No need to justify your answer.)

## Applied/Numerical Analysis Qualifying Exam

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**Cover Sheet – Numerical Analysis Part**

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**Combined Applied Analysis/Numerical Analysis Qualifier**
**Numerical analysis part**
**August, 2011**

In all questions below, $\Omega$ is a bounded polygonal domain with boundary $\partial\Omega$ and $\mathcal{T}_h$ is a regular family of triangulations of $\overline{\Omega}$.

**Problem 1.** Let $\mathbb{P}_2$ be the space of polynomials in two variables spanned by $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, let $\hat{T}$ be the reference unit triangle, $\hat{\gamma}$ one side of $\hat{T}$, and $\hat{\pi}$ the standard Lagrange interpolant in $\hat{T}$ with values in $\mathbb{P}_2$.

Recall that there exists a constant $C$ only depending on the geometry of $\hat{T}$ such that

$$\forall v \in H^3(\hat{T}),\ \inf_{p\in\mathbb{P}_2} \|v+p\|_{H^3(\hat{T})} \leq C|v|_{H^3(\hat{T})}.$$

(a) State the trace theorem relating $L^2(\hat{\gamma})$ and $H^1(\hat{T})$.

(b) Prove that there exists a constant $\hat{C}$ only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that

$$\forall \hat{u} \in H^3(\hat{T}),\ \|\hat{u} - \hat{\pi}(\hat{u})\|_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\hat{T})}.$$

(c) Let

$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\ \forall T \in \mathcal{T}_h\,,\ v_h|_T \in \mathbb{P}_2\}.$$

Let $T$ be a triangle of $\mathcal{T}_h$ with diameter $h_T$ and diameter of inscribed disc $\varrho_T$, and let $\gamma$ be one side of $T$. Let $F_T$ be the affine mapping from $\hat{T}$ onto $T$ and let $\pi_{2,h}$ denote the standard Lagrange interpolant on $X_h$. Prove that there exists a constant $C$ only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that

$$\forall u \in H^3(T),\ \|u - \pi_{2,h}(u)\|_{L^2(\gamma)} \leq C\sigma_T h_T^{2+1/2}|u|_{H^3(T)},$$

where $\sigma_T = h_T/\varrho_T$.

**Problem 2.** Let $\delta > 0$ be a given constant parameter and $u \in H_0^1(\Omega)$ a given function. Consider the problem: Find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\delta^2\Delta\,\varphi^\delta(x) + \varphi^\delta(x) &= u(x) \text{ a.e. in } \Omega\,, \\ \varphi^\delta(x) &= 0 \text{ a.e. on } \partial\Omega. \end{aligned} \tag{2.1}$$

(a) Define the bilinear form

$$a_\delta(u,v) = \delta^2\int_\Omega \nabla\,u(x)\cdot\nabla\,v(x)dx + \int_\Omega u(x)v(x)dx.$$

Write the variational formulation of Problem (2.1) and prove that it has one and only one solution $\varphi^\delta \in H_0^1(\Omega)$.

(b) Prove that

$$\|\varphi^\delta\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

(c) Prove that

$$\|\nabla\,\varphi^\delta\|_{L^2(\Omega)} \leq \|\nabla\,u\|_{L^2(\Omega)}.$$

Hint: observe that $\Delta\,\varphi^\delta$ belongs to $L^2(\Omega)$, take the scalar product of (2.1) with $-\Delta\,\varphi^\delta$ and apply Green's formula.

(d) Now let

$$X_{0,h} = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall T \in \mathcal{T}_h\,,\, v_h|_T \in \mathbb{P}_1\,,\, v_h|_{\partial\Omega} = 0\}.$$

Given $u_h$ in $X_{0,h}$, consider the discrete problem: Find $\varphi_h^\delta \in X_{0,h}$ satisfying

$$\forall v_h \in X_{0,h}\,,\; a_\delta(\varphi_h^\delta, v_h) = \int_\Omega u_h(x) v_h(x) dx\,. \tag{2.2}$$

(i) Prove that problem (2.2) has has one and only one solution $\varphi_h^\delta \in X_{0,h}$.

(ii) Prove that

$$\|\varphi_h^\delta\|_{L^2(\Omega)} \le \|u_h\|_{L^2(\Omega)}.$$

(e) Assume that $\varphi^\delta$ belongs to $H^2(\Omega)$. Let $\pi_{1,h}$ denote the standard Lagrange interpolant on $X_{0,h}$.

(i) Prove that

$$a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) = a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) - \int_\Omega (u - u_h)\big(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta)\big) dx\,.$$

(ii) Assuming that $u$ is smooth enough, $u_h = \pi_{1,h}(u)$, and $\delta = h$, derive an estimate for $\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}$.

**Problem 3.** Let $T > 0$ be a given final time, let $\vec{b}$ be a given vector valued function with components in $L^2(0,T;H^1(\Omega)) \cap \mathcal{C}^0(\overline{\Omega} \times [0,T])$ and let $u_0$ be a given real valued function in $\mathcal{C}^0(\overline{\Omega})$. We suppose that

$$\operatorname{div} \vec{b} = 0 \text{ a.e. in } \Omega\,,\; \vec{b} = \vec{0} \text{ on } \Gamma\,.$$

Consider the time-dependent problem: Find $u$ such that

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) + \vec{b}(x,t)\cdot\nabla\, u(x,t) &= 0 \text{ a.e. in } \Omega\times]0,T[\,, \\ u(x,0) &= u_0(x) \text{ a.e. in } \Omega\,, \end{aligned} \tag{3.1}$$

where

$$\vec{b}\cdot\nabla\, u = b_1\frac{\partial u}{\partial x_1} + b_2\frac{\partial u}{\partial x_2}.$$

Accept as a fact that (3.1) has one and only one solution $u$ that is sufficiently smooth. It is discretized as follows in space and time. Let

$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall T \in \mathcal{T}_h\,,\, v_h|_T \in \mathbb{P}_1\}.$$

Choose an integer $K \ge 2$, set $k = T/K$, $t_n = nk$ and $u_h^0 = \pi_{1,h}(u_0)$. For $1 \le n \le K$, define $u_h^n \in X_h$ from $u_h^{n-1}$ recursively by

$$\forall v_h \in X_h\,,\; \frac{1}{k}\int_\Omega (u_h^n - u_h^{n-1})(x) v_h(x) dx + \int_\Omega (\vec{b}(x,t_n)\cdot\nabla\, u_h^n(x)) v_h(x) dx = 0\,. \tag{3.2}$$

(a) Prove that

$$\forall v_h \in X_h\,,\; \int_\Omega (\vec{b}(x,t_n)\cdot\nabla\, v_h(x)) v_h(x) dx = 0.$$

(b) Show that, given $u_h^{n-1} \in X_h$, (3.2) has one and only one solution $u_h^n$ in $X_h$.

(c) Prove for $1 \le n \le K$

$$\|u_h^n\|_{L^2(\Omega)} \le \|u_h^0\|_{L^2(\Omega)}.$$

(d) Is the matrix of the system (3.2) symmetric ? Justify your answer.