

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 13, 2025

Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Problem 1. Let $x_j = j/n$, $j = 0, \dots, n$. Let $S_0^{1/n}(3, 1) = \{s \in S_n^{1/n}(3, 1) : s(0) = s(1) = 0\}$. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x| - 1)^2(2|x| + 1) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}, \quad \psi(x) = \begin{cases} x(|x| - 1)^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

These satisfy $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx - j)$ satisfy $\phi_j(k/n) = \delta_{j,k}$, $\phi'_j(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi'_j(k/n) = \delta_{j,k}$.

- (a) Show that $\langle u, v \rangle = \int_0^1 u''v''dx$ defines an inner product on $S_0^{1/n}(3, 1)$, and that $\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^n$ is a basis for $S_0^{1/n}(3, 1)$.
- (b) Suppose that $s \in S_0^{1/n}(3, 1)$ and that s minimizes $\int_0^1 (s''(x))^2 dx$, given $s(x_j) = f_j$ for $j = 1, \dots, n-1$. Show that $s \in C^{(2)}[0, 1]$.

Problem 2. Consider the operator $Lu = xu'' - u'$ with domain $\mathcal{D}_L := \{u \in L^2[1, 2] : Lu \in L^2[1, 2], u(1) = 0 \text{ \& } u'(2) = 0\}$. You are given that the homogenous solutions of $Lu = 0$ are 1 and x^2 , neither of which is in $\mathcal{D}_L$.

- (a) Compute the adjoint L^* , along with the adjoint boundary conditions, i.e. find $\mathcal{D}_L^*$. Is L self adjoint?
- (b) Compute the Green's function for L . Is L^{-1} compact? Justify your answer.

Problem 3. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

- (a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
- (b) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = x^2\phi(x)$ for some $\phi \in \mathcal{D}$ if and only if $\psi(0) = 0$ and $\psi'(0) = 0$.
- (c) Use (b) to find all $T \in \mathcal{D}'$ that satisfy $x^2T(x) = 0$.

Problem 4. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$.

- (a) State and prove the Fredholm alternative and state the Closed Range Theorem.
- (b) Let $\mathcal{H}$ be $L^2[0, 1]$ and consider the finite-rank kernel $k(x, y) = \sum_{j=1}^n \phi_j(x)\overline{\psi_j(y)}$ and its associated operator $Ku(x) = \int_0^1 k(x, y)u(y)dy$. Both $\{\phi_j\}_{j=1}^n$ and $\{\psi_j\}_{j=1}^n$ are linearly independent subsets of $L^2[0, 1]$. Show that K is a Hilbert-Schmidt operator.
- (c) Consider the operator $L = I - \lambda K$. Find all values of λ such that $Lu = f$ has a unique solution for all $f \in L^2[0, 1]$. (Hint: these are related to the eigenvalues of a matrix.)

NUMERICAL ANALYSIS QUALIFIER

August 13, 2025

Problem 1. Let $T = [0, 1] \times [0, 1]$ be the unit square in $\mathbb{R}^2$, with vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, and $v_4 = (0, 1)$. Let $Q_1 = \{a + bx + cy + dxy : a, b, c, d \in \mathbb{R}\}$ be the bilinear functions on T , i.e., the functions that are linear in each coordinate direction. Finally, let $\sigma_i(u) = u(v_i)$, $i = 1, 2, 3, 4$, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$.

- (a) Show that (T, Q_1, Σ) is a finite element triple.
- (b) Given $u \in C(\overline{T})$, let $I_h u \in Q_1$ satisfy $(I_h u)(v_i) = u(v_i)$, $i = 1, \dots, 4$. Show that I_h is stable on $C(\overline{T})$, that is,

$$\|I_h u\|_{L_\infty(T)} \leq \|u\|_{L_\infty(T)}.$$

(It may help to recall that members of Q_1 are linear in each coordinate direction.)

Also, show that I_h is a projection on Q_1 in the sense that $I_h q = q$ whenever $q \in Q_1$.

- (c) Show that

$$\|u - I_h u\|_{L_2(T)} \leq C|u|_{H^2(T)}, \quad u \in H^2(T),$$

where C is independent of u .

Note: You may use standard analysis results such as trace, Sobolev, and Poincaré inequalities and the Bramble-Hilbert Lemma without proof, but specify precisely which results you are using.

Problem 2. Consider the problem

$$\begin{aligned} u'''' - (k(x)u')' + q(x)u &= f \text{ in } (0, 1), \\ u(0) &= u'(0) = 0, \\ u(1) &= 0, \quad u''(1) = \gamma. \end{aligned}$$

Here $f(x)$, $k(x) \geq 0$, $q(x) \geq 0$, and γ are given data.

- (a) Derive the weak formulation of this problem. Specify the appropriate Sobolev space for the weak formulation.
- (b) Suggest a finite element approximation to this problem using conforming piecewise polynomial functions over a uniform partition of Ω into subintervals with length h .
- (c) Derive an error estimate for the finite element solution in the energy norm corresponding to the variational (Sobolev) space from part (a). You may assume coercivity of the bilinear form that you defined in part (a) along with typical approximation properties for the finite element space you defined in part (b).

Problem 3. Consider the following initial boundary value problem: find a solution $u(x, t)$ such that

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \alpha u = f, & \text{for } x \in \Omega, \ 0 < t \leq T, \\ u(x, t) = 0, & \text{for } x \in \partial\Omega, \ 0 < t \leq T, \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, $\partial\Omega$ is its boundary, $\alpha \in \mathbb{R}$ is a given constant, and $f(x, t)$ and $u_0(x)$ are given right hand side and initial data functions.

In the following let $V = H_0^1(\Omega)$ and let $V_h \subset V$ be a finite element approximation space with (nodal) basis $\varphi_i^h(x)$, where V_h consists of piecewise polynomials of degree r defined with respect to a shape regular and quasi-uniform triangular mesh of diameter h . You may use standard Poincaré inequalities without proof.

- (a) Define the bilinear form $a(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + \alpha uv) dx.$$

Show that there exists $\alpha_0 < 0$ such that a is coercive on $H_0^1(\Omega)$ (with the standard norm $\|u\|_{H^1(\Omega)} = (\int_{\Omega} (|\nabla u|^2 + u^2) dx)^{1/2}$) whenever $\alpha > \alpha_0$.

- (b) Define a semidiscrete finite element approximation $u_h(t) : [0, T] \rightarrow V_h$ of u based on the finite element space V_h .
 (c) Given the finite element approximation $u_h(t)$ defined above, prove that if $\alpha > \alpha_0$ there are $c_1, c_2 > 0$ such that

$$\|u_h(T)\|_{L_2(\Omega)}^2 + c_1 \int_0^T \|u_h(s)\|_{H^1(\Omega)}^2 \leq \|u_h(0)\|_{L_2(\Omega)}^2 + c_2 \int_0^T \|f(t)\|_{L_2(\Omega)}^2.$$

- (d) State and prove an optimal-order error estimate for $\|(u - u_h)(T)\|_{L_2(\Omega)}$. You may assume an H^2 regularity property for the Dirichlet problem for the elliptic operator $-\Delta + \alpha$, and you may also use results from elliptic finite element theory and finite element approximation theory without proof.