Applied/Numerical Analysis Qualifying Exam

January 6, 2014

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
January 6, 2014

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let f be a continuous, 2π periodic function having the Fourier series $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$. The trapezoidal rule for numerically finding $\int_0^{2\pi} f(t)dt$ is given by

$$Q_n(f) = \frac{2\pi}{n}\sum_{k=0}^{n-1} f(2\pi k/n).$$

(a) Let $S_m(t) = \sum_{k=-m}^{m} c_k e^{ikt}$. Show that $Q_n(S_{n-1}) = \int_0^{2\pi} f(t)dt$.

(b) Show that $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right| \le 2\pi \|f - S_{n-1}\|_{C[0,2\pi]}$.

(c) Suppose that $|c_k| \le |k|^{-6}$ for all $k \ne 0$. Estimate $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right|$.

Problem 2. Consider the Sturm-Liouville (S-L) problem

$$u'' = f,\ u'(0) = 0,\ u(1) + u'(1) = 0.$$

(a) Find the Green's function, $G(x, y)$, for this problem.

(b) Show that $Gf(x) = \int_0^1 G(x,y)f(y)dy$ is compact and self adjoint on $L^2[0,1]$.

(c) Show that the eigenfunctions of the eigenvalue problem $u'' + \lambda u$, $u'(0) = 0$, $u(1) + u'(1) = 0$ form a complete set orthogonal set in $L^2[0,1]$. (Hint: Show that the the null space of G is $\{0\}$.)

Problem 3. Find the first term of the asymptotic series for $F(x) := \int_0^\infty e^{xt - \frac{1}{2}t^2} dt$, $x \to +\infty$.

Problem 4. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. In addition, let the Fourier and inverse Fourier transforms be given by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \text{ and } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

(a) Define $\mathcal{S}$ and give the semi-norm topology for it. In addition, define $\mathcal{S}'$.

(b) Given that $\mathcal{F}$ is a continuous bijection mapping $\mathcal{S} \to \mathcal{S}$, define the Fourier transform of a tempered distribution.

(c) Show that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$.

(d) Let $T(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$. Show that $T'(t) = \delta(t+1) - \delta(t-1)$. Use (c) to find $\widehat{T}$.

Applied/Numerical Analysis Qualifying Exam

January 6, 2014

Cover Sheet – Numerical Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

NUMERICAL ANALYSIS QUALIFIER

January, 2014

Problem 1. In this problem $\mathbb{P}^j$ denotes the space of polynomials on $\mathbb{R}^2$ of degree at most j. Let Ω be a polygonal domain with boundary Γ in $\mathbb{R}^2$, f a given function in $L^2(\Omega)$, and $u \in H^1(\Omega)$ the solution of

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma uv ds = \int_\Omega fv dx =: L(v), \qquad \text{for all } \ v \in H^1(\Omega). \tag{1.1}$$

Let $\mathcal{T}_h$, $0 < h < 1$, be shape regular triangulations of Ω. The elements of this partitioning will be denoted by τ and and the edges of the elements are denoted by e. Set

$$V_h := \{v_h \in H^1(\Omega) \ : \ v_h|_\tau \in \mathbb{P}^1, \quad \tau \in \mathcal{T}_h\}$$

(equipped with the norm in $H^1(\Omega)$) and the composite trapezoidal quadrature

$$Q_h(w) := \sum_{e \subset \Gamma} \frac{|e|}{2}(w(P_1) + w(P_2)),$$

where P_1 and P_2 are the end points of the edge e. Consider the following Galerkin FEM: find $u_h \in V_h$ such that

$$a_h(u_h, \phi) := (\nabla u_h, \nabla \phi) + Q_h(u_h \phi) = L(\phi), \quad \forall \phi \in V_h.$$

(a) Derive the strong form of problem (1.1) assuming that the solution u is smooth.

(b) Consider as given the coercivity of $a(.,.)$ in H^1, i.e. for some constant $c_0 > 0$ and for all $u \in H^1(\Omega)$ we have $a(u,u) \geq c_0 \|u\|^2_{H^1(\Omega)}$. Show that there is a constant $\alpha_0 > 0$, independent of h, such that

$$Q_h(v^2) \geq \alpha_0 \int_\Gamma v^2 ds \quad \forall v \in V_h \quad .$$

and deduce the (uniform) coercivity of $a_h(\cdot,\cdot)$ in V_h.

(c) Prove that there is a constant $\alpha_1 > 0$, independent of h, satisfying

$$\|u_h - u\|_{H^1(\Omega)} \leq \alpha_1 \left\{ \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} + \sup_{w_h \in V_h} \frac{|a(u, w_h) - a_h(u, w_h)|}{\|w_h\|_{H^1(\Omega)}} \right\}.$$

Problem 2. Given $T > 0$, consider the following parabolic initial boundary value problem for $u(x,t)$:

$$\begin{aligned} &u_t(x,t) - u_{xx}(x,t) + u(x,t) = 0, \quad \text{for } \ x \in (0,1), \ 0 < t \leq T, \\ &u(0,t) = 0, \ u_x(1,t) + u(1,t) = g(t), \ 0 < t \leq T, \\ &u(x,0) = u_0(x), \ x \in (0,1), \end{aligned} \tag{2.1}$$

where $g(t)$ is a given function of $t \in (0,T]$ and $u_0(x)$ is a given function of $x \in (0,1)$.

(a) Consider the following weak form of (2.1): find $u(x,t)$ such that for any fixed $t > 0$, $u(t) \in V$ and satisfies

$$\int_0^1 u_t(x,t)\phi(x)dx + a(u(t), \phi) = L(t, \phi), \qquad 0 < t \leq T, \quad \forall \phi \in V,$$

What are the space V, the bilinear form $a(\cdot,\cdot)$, and the linear form $L(t,\cdot)$ corresponding to the above problem?

(b) Consider a partition of the interval $(0, 1)$ into N equal elements with size $h = 1/N$. Let $V_h \subset V$ be the finite element space of continuous piecewise linear functions over the partitioning. Introduce the semi-discrete Galerkin method and compute the global stiffness matrix matrix corresponding to the bilinear form $a(\cdot, \cdot)$ for $N = 2$.

(c) Given an integer $M > 0$, define $k = T/M$ and $t_n = kn$, $n = 0, 1, 2, \ldots, M$. Let $U_h^n \in V_h$ be an approximation of $u(., t_n)$ obtained by the fully implicit scheme in time (we take $g(t) = 0$):

$$\int_0^1 (U_h^n(x) - U_h^{n-1}(x))\,\phi(x)dx + k\; a(U_h^n, \phi) = 0, \;\; \forall \phi \in V_h,\; n = 1, 2, \ldots, M,$$

with U_h^0 determined through the initial data. Derive an *a priori* estimate for $\max_{n=1,\ldots,M} \|U_h^n\|_V$ through U_h^0.

Problem 3. Consider the unit segment $K = [0, 1]$ and let P be the set of functions that are piecewise quadratic over the intervals $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ and are of class $\mathbb{C}^1$ over K. Functions in P are continuous and their first derivatives are continuous at $\frac{1}{2}$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_4\}$ be defined for $p \in P$ as $\sigma_1(p) = p(0)$, $\sigma_2(p) = p'(0)$, $\sigma_3(p) = p(1)$, $\sigma_4(p) = p'(1)$. Prove that the triple (K, P, Σ) is a finite element. Compute the shape functions associated with the degrees of freedom in Σ.