APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 9, 2020

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Consider $F(x) := \frac{x}{2} + \frac{1}{x}$, $1 \le x \le 2$.

(a) State and prove the Contraction Mapping Theorem.
(b) Show that $F : [1,2] \to [1,2]$, that it is Lipschitz continuous on $[1,2]$, with Lipschitz constant less than or equal to $1/2$.
(c) Obviously, the fixed point is $\sqrt{2}$. If $x_0 = 2$, estimate the number of iterations needed to come within 0.001 of $\sqrt{2}$.

Problem 2. Let $p \in C^{(2)}[0,1], q \in C[0,1]$ be positive on $[0,1]$. Consider the operator $Lu = -(pu')' + qu$, where $\mathcal{D}_L := \{u \in L^2[0,1]] : Lu \in L^2[0,1]],\ u(0) = 0\ \&\ u'(1) = 0\}$.

(a) Show that L is self adjoint and positive definite.
(b) Explain why the Green's function $g(x,y)$ exists for this problem.
(b) Prove that the eigenfunctions of L contain a complete, orthonormal set with respect to $L^2[0,1]$.

Problem 3. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{C}(\mathcal{H})$ the compact operators $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$.

(a) Prove that $\mathcal{C}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$.
(b) Let $\mathcal{H} = L^2[0,1]$. Use the result above to show that a Hilbert-Schmidt operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$, $k \in L^2([0,1] \times [0,1])$ is compact.

Problem 4. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. The Fourier transform convention used here is $\mathcal{F}[f](\omega) = \widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{i\omega t}dt$, $\mathcal{F}^{-1}[\widehat{f}](x) = f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)e^{-i\omega t}d\omega$.

(a) Sketch a proof: *The Fourier transform $\mathcal{F}$ is a continuous linear operator mapping $\mathcal{S}$ into itself.*
(b) Use the previous result to show that[1] $\langle \mathcal{F}[T](x), \phi(x)\rangle := \langle T(x), \mathcal{F}[\phi](x)\rangle$ implies $\mathcal{F}[T] \in \mathcal{S}'$.
(c) You are **given** that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$. Let T be the tent function $T(x) = 1 - |x|$, $|x| \le 1$, and $T(x) = 0$ otherwise. Find $\widehat{T}$. (Hint: What is T''?)

[1] Here we are defining $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$. Note that there is no complex conjugate in this definition of $\langle f, g\rangle$.

NUMERICAL ANALYSIS QUALIFIER

January, 2020

Problem 1. Consider the following two finite elements: (τ, Q_1, Σ) and $(\tau, \widetilde{Q}_1, \Sigma)$, where

$$\begin{aligned}
&\tau = [-1,1]^2\\
&Q_1 = span\{1, x, y, xy\},\\
&\widetilde{Q}_1 = span\{1, x, y, x^2 - y^2\}\\
&\Sigma = \{w(-1,0), w(1,0), w(0,-1), w(0,1)\}.
\end{aligned}$$

Obviously, Σ is the set of the values of a function $w(x, y)$ at the midpoints of the edges of τ.

(a) Show that the finite element (τ, Q_1, Σ) is not unisolvent.
(b) Show that the finite element $(\tau, \widetilde{Q}_1, \Sigma)$ is unisolvent.
(c) Show that the finite element spaces are in general not H^1-conforming.

Problem 2. Consider the boundary value problem

$$\begin{aligned}
&u^{(4)}(x) + q(x)u = f(x), \qquad 0 < x < 1,\\
&u(0) = 0,\ u(1) = 0,\\
&u''(0) = -\gamma,\ u'(1) + u''(1) = \beta,
\end{aligned} \tag{2.1}$$

where $f(x)$ is a given function on $(0, 1)$, β and γ are given constants and $q(x) \geq 0$.

(a) Give a weak formulation of this problem in an appropriate space V, characterize V, and prove that the corresponding bilinear form is coercive on V.

(b) Set up a finite dimensional space $V_h \subset V$ of piece-wise cubic functions over a uniform partition of $(0, 1)$. Introduce the Galerkin finite element method for the problem (2.1) for V_h. State an error estimate in V-norm assuming that $u(x) \in H^4(0, 1)$ (do NOT prove this).
(c) Assuming "full regularity" and using duality argument **prove** the following estimate for the error of the Galerkin solution u_h:

$$\|u - u_h\|_{L^2} \leq Ch^4 \|u^{(4)}\|_{L^2}. \tag{2.2}$$

Further prove the estimate $\|u' - u_h'\|_{L^2} \leq Ch^3 \|u^{(4)}\|_{L^2}$.

Problem 3. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, and let $\mathcal{T}_h$ be a shape-regular and quasi-uniform triangulation of Ω with element diameters uniformly equivalent to h. Let also $V_h \subset H_0^1(\Omega)$ be a piecewise linear Lagrange finite element space. You may assume the existence of an interpolation operator $I_h : H_0^1(\Omega) \to V_h$ satisfying

$$\|u - I_h u\|_{L_2(\Omega)} + h\|u - I_h u\|_{H^1(\Omega)} \leq Ch^2 |u|_{H^2(\Omega)}.$$

(a) Let $u(t) \in H_0^1(\Omega)$ $(0 \leq t \leq T)$, u_0, and f be sufficiently smooth such that

$$\int_\Omega u_t v\ dx + \int_\Omega \nabla u \cdot \nabla v\ dx = \int_\Omega f v\ dx,\ v \in H_0^1(\Omega),\ 0 < t \leq T,$$
$$u(x, 0) = u_0(x),\ x \in \Omega.$$

Write down the spatially semidiscrete (i.e., discretized in space but not in time) finite element formulation of this problem. Denote by u_h the solution to these finite element equations.
(b) For $0 < t \leq T$, let now $\tilde{u}_h(t)$ be the *elliptic* finite element approximation to $u(t)$. That is

$$\int_\Omega \nabla \tilde{u}_h(t) \cdot \nabla v_h\ dx = \int_\Omega \nabla u(t) \cdot \nabla v_h\ dx,\ v_h \in V_h.$$

Prove that

$$\int_\Omega (u_h - \tilde{u}_h)_t v_h \; dx + \int_\Omega \nabla(u_h - \tilde{u}_h) \cdot \nabla v_h \; dx = \int_\Omega (u - \tilde{u}_h)_t v_h \; dx, \; v_h \in V_h, \; 0 < t \leq T.$$

(c) Next recall Gronwall's Lemma, which states that if σ and ρ are continuous real functions with $\sigma \geq 0$ and $c \geq 0$ is a constant, and if

$$\sigma(t) \leq \rho(t) + c \int_0^t \sigma(s) \; ds, \; t \in [0,T],$$

then

$$\sigma(t) \leq e^{ct} \rho(t), \; t \in [0,T].$$

Using this result, prove that

$$\|(u_h - \tilde{u}_h)(T)\|_{L_2(\Omega)}^2 \leq C(T) \left(\|(u_h - \tilde{u}_h)(0)\|_{L_2(\Omega)}^2 + \int_0^T \|(u - \tilde{u}_h)_t(s)\|_{L_2(\Omega)}^2 \; ds \right).$$

(d) For the final part you will need the following intermediate result. Given $v \in H_0^1(\Omega) \cap H^2(\Omega)$, let $v_h \in V_h$ satisfy

$$\int_\Omega \nabla v_h \cdot \nabla w_h dx = \int_\Omega \nabla v \cdot \nabla w_h dx, \quad \text{all } w_h \in V_h.$$

Then

$$\|v - v_h\|_{L_2(\Omega)} \leq C h^2 |v|_{H^2(\Omega)}.$$

Assuming this result and additionally that $\|(u - u_h)(0)\|_{L_2(\Omega)} \leq C h^2 |u(0)|_{H^2(\Omega)}$, prove that

$$\|(u - u_h)(T)\|_{L_2(\Omega)} \leq C(T) h^2 \left(|u(0)|_{H^2(\Omega)} + \left(\int_0^T |u_t|_{H^2(\Omega)}^2 \right)^{1/2} \right).$$