APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 11, 2021

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n}\sum_{j=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

(a) State and prove the Convolution Theorem for the DFT.

(b) Let a, x, y be column vectors with entries $a_0, \ldots, a_{n-1}$, $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$. In addition, let α, ξ and η be n-periodic sequences, the entries for one period, $k = 0, \ldots, n-1$, being those of a, x, and y, respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

(c) What are the eigenvalues and eigenvectors of A? Use parts (a) and (b) to prove your answer.

Problem 2. Let $Lu = -\frac{d^2u}{dx^2}$, $0 \le x \le 1$, with the domain of L given by

$$D_L := \{u \in L^2[0,1] : u'' \in L^2[0,1],\ u(0) = -u(1),\ u'(0) = -u'(1)\}.$$

(a) Show that L is self adjoint on $D(L)$.

(b) Find the Green's function $G(x,y)$ for the problem $Lu = f$, $u \in D_L$.

(c) Show that $Ku := \int_0^1 G(\cdot, y)u(y)dy$ is a compact self-adjoint operator.

(d) Without actually finding them, show that the eigenfunctions of L contain an orthonormal set that is complete in $L^2[0,1]$.

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Closed Range Theorem.

(b) Let $\mathcal{H} = L^2[0,1]$. Define the kernel $k(x,y) := x^3y^2$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show the K is in $\mathcal{C}(L^2[0,1])$.

(c) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (b) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0,1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \le 1 \\ 0 & |x| > 1 \end{cases}, \qquad \psi(x) = \begin{cases} x(|x|-1)^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

(a) Let $n \ge 2$ and $0 \le j \le n$. Show that the functions $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx-j)$ satisfy the following: $\phi_j(k/n) = \delta_{j,k}$, $\phi_j'(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi_j'(k/n) = \delta_{j,k}$.

(b) Use part (a) to show that the set $\{\phi_j, \psi_j\}_{j=0}^n$ forms a basis for the finite element space $S^{\frac{1}{3}}(3,1)$. You may assume that $\dim S^{\frac{1}{3}}(3,1) = 2n+2$.

(c) Use part (b) to define a (non-orthogonal) projection on $C^{(1)}[0,1]$.

Numerical Analysis Part

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Problem 1. Let $\widehat{K} \subset \mathbb{R}^2$ be the reference triangle with vertices $\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0)$, and $\widehat{\boldsymbol{z}}_3 = (0,1)$. Let $\widehat{S} \subset \mathbb{R}$ be the reference edge with end points $\widehat{\boldsymbol{a}}_1 = 0$ and $\widehat{\boldsymbol{a}}_2 = 1$. Let $k \in \mathbb{N}$. Let $\mathbb{P}_{k,d}$ denote the real vector space composed of the d-variate polynomials of degree at most k.

1. Let $\widehat{\mu}_1$, $\widehat{\mu}_2 \in \mathbb{P}_{1,1}$ be the barycentric coordinates associated with the vertices $\widehat{\boldsymbol{a}}_1$, $\widehat{\boldsymbol{a}}_2$, respectively. Give the expressions of $\widehat{\mu}_1(\widehat{x})$, $\widehat{\mu}_2(\widehat{x})$, (no proof needed).

2. Let $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, $\widehat{\lambda}_3 \in \mathbb{P}_{1,2}$ be the barycentric coordinates associated with the vertices $\widehat{\boldsymbol{z}}_1$, $\widehat{\boldsymbol{z}}_2$, $\widehat{\boldsymbol{z}}_3$, respectively. Give the expressions of $\widehat{\lambda}_1(\widehat{x},\widehat{y})$, $\widehat{\lambda}_2(\widehat{x},\widehat{y})$, $\widehat{\lambda}_3(\widehat{x},\widehat{y})$ (no proof needed).

3. For all $i \in \{1,2,3\}$, let $\widehat{E}_i$ be the edge of $\widehat{K}$ with the endpoints $\widehat{\boldsymbol{z}}_i$ and $\widehat{\boldsymbol{z}}_{i+1}$, with the convention that $\boldsymbol{z}_4 := \boldsymbol{z}_1$. Give the expression of the unique affine geometric mapping $\boldsymbol{T}_{\widehat{E}_i} : \widehat{S} \to \mathbb{R}$ that maps $\widehat{S}$ to $\widehat{E}_i$ and is such that $\boldsymbol{T}_{\widehat{E}_i}(\widehat{\boldsymbol{a}}_1) = \widehat{\boldsymbol{z}}_i$.

4. For all $i \in \{1,2,3\}$, what is the size of the Jacobian matrix of $\boldsymbol{T}_{\widehat{E}_i} : \widehat{S} \to \mathbb{R}$ (i.e., how many rows and columns)? Compute the Jacobian matrix.

5. Let $K \subset \mathbb{R}^2$ be a triangle with vertices $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$ (all assumed to be distinct). How many affine geometric transformations there are that map $\widehat{K}$ to K?

6. Give the expression of the unique affine geometric mapping $\boldsymbol{T}_K : \widehat{K} \to \mathbb{R}^2$ that maps $\widehat{K}$ to K and is such that $\boldsymbol{T}_K(\widehat{\boldsymbol{z}}_i) = \boldsymbol{z}_i$ for all $i \in \{1,2,3\}$.

7. Let $\widehat{P}_{\widehat{K}} := \{\widehat{q}_{|\widehat{K}}, q \in \mathbb{P}_{2,2}\}$ (i.e., $\widehat{P}$ is composed of the restrictions to $\widehat{K}$ of the two-variate polynomials of degree at most 2). For all $i \in \{1,2,3\}$, let $\widehat{\sigma}_i^{\mathrm{v}} \in \mathcal{L}(\mathbb{P}_{1,2};\mathbb{R})$ be defined by setting $\widehat{\sigma}_i^{\mathrm{v}}(\widehat{p}) := \widehat{p}(\widehat{\boldsymbol{z}}_i)$. Let $\widehat{\sigma}_i^{\mathrm{e}} \in \mathcal{L}(\widehat{P};\mathbb{R})$ be defined by setting $\widehat{\sigma}_i^{\mathrm{e}}(\widehat{p}) := \frac{1}{|\widehat{E}_i|}\int_{\widehat{E}_i} \widehat{p}\,\mathrm{d}l$, where $|\widehat{E}_i|$ is the length of $\widehat{E}_i$. Let $\widehat{\Sigma} := \{\widehat{\sigma}_1^{\mathrm{v}}, \widehat{\sigma}_2^{\mathrm{v}}, \widehat{\sigma}_3^{\mathrm{v}}, \widehat{\sigma}_1^{\mathrm{e}}, \widehat{\sigma}_2^{\mathrm{e}}, \widehat{\sigma}_3^{\mathrm{e}}\}$. Prove that $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is a unisolvent finite element.

Problem 2. Let $D := (0,1)$. Let $V := \{v \in H^1(D) \mid v(0) = 0\}$ equipped with the inner product $\int_D (u'(x)v'(x) + u(x)v(x))\,\mathrm{d}x$. Accept as a fact that V is a Hilbert space. Let $u_0, u_1 \in \mathbb{R}$ and $f \in C^0(D;\mathbb{R})$. Consider the following two-point boundary value problem:

$$
\begin{aligned}
-u''(x) + u(x) &= f(x), \quad x \in D, \\
u(0) &= u_0, \\
u'(1) + u(1) &= u_1.
\end{aligned}
\tag{1}
$$

1. Write a weak formulation of this problem.

2. Prove that $|v(1)| \leq \|v'\|_{L^2(D)}$ for all $v \in V$. (*Hint*: Use without proving it that $W := \{v \in C^1(D;\mathbb{R}) \mid v(0) = 0\}$ is dense in V.)

3. Prove that the proposed weak formulation is well-posed. (Prove in details that all the assumptions of the theoretical result you invoke are met.)

Problem 3. Consider the problem stated in (1). The purpose of this problem is to construct a finite difference approximation of (1). Let u be the solution to (1) and assume that u has four continuous derivatives on the closed interval $[0, 1]$. Let N be a nonzero natural number. Let $h := \frac{1}{N}$ and $x_i := ih$, for $i = 0, \ldots, N$. Let us set $f_i := f(x_i)$ for all $i \in \{0, \ldots, N\}$. The finite difference approximation of (1) we consider consists of seeking $(y_i)_{i\in\{0,\ldots,N\}} \in \mathbb{R}^{N+1}$ so that

$$\begin{aligned} y_0 &= u_0 \\ -\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i &= f_i, \qquad i = 1, \ldots, N-1, \\ \frac{y_N - y_{N-1}}{h} + y_N &= u_1. \end{aligned}$$

1. Let $i \in \{1, \ldots, N-1\}$. Using Taylor expansions, compute

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} - u''(x_i).$$

2. Using Taylor expansions, compute $\frac{u(x_N)-u(x_{N-1})}{h} - u'(x_N)$.

3. Prove the following a priori estimate:

$$\max_{0\le j\le N} y_j \le \max\{u_0, u_1\} + \max_{1\le j\le N-1} f(x_j).$$

(*Hint:* If $y_i = \max_{0\le j\le N} y_j$, then $y_i - y_{i+1} \ge 0$ and $y_i - y_{i-1} \ge 0$. Notice also that $-y_{i-1} + 2y_i - y_{i+1} = y_i - y_{i-1} + y_i - y_{i+1}$. Distinguish three cases: the maximum is attained at $i = 0$, at $i \in \{1, \ldots, N-1\}$, or at $i = N$.)

4. Prove the following a priori estimate:

$$\max_{0\le j\le N} |y_i| \le \max\{|u_0|, |u_1|\} + \max_{1\le j\le N-1} |f(x_i)|.$$

(*Hint:* reason as above to derive an estimate on $\min_{0\le i\le N} y_j$ and conclude.)

5. Introduce the error $e_i := y_i - u(x_i)$ and show that

$$\max_{0\le i\le N} |e_i| \le \frac{h}{2}\, \max\{\max_{0\le x\le 1} |u''(x)|, \frac{h}{6} \max_{0\le x\le 1} |u^{(4)}(x)|\}.$$