Complex analysis qualifying exam, August 2012.

1. Give the statements of the following:

   (a) Runge's theorem;
   (b) the Schwarz lemma.

2. Find the Laurent expansion for the function $z^{-1}(z+1)^{-1}$ centered at $a=1$ in $\{1<|z-1|<2\}$.

3. Let f be a meromorphic function in $\mathbb{C}$. Suppose that f is doubly periodic, i.e. that for some non-zero numbers $a, b \in \mathbb{C}$,

$$f(z)=f(z+a) \text{ and } f(z)=f(z+b)$$

for any $z \in \mathbb{C}$. Consider the parallelogram P with sides $(0,a)$ and $(0,b)$. Assuming that f does not have any zeros or poles on the sides of P, prove that the number of zeros of f in P is equal to the number of poles of f in P.

4. Apply the Residue Theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+1)(x^2+4)}.$$

5. Let $u(z)$ be a real continuous function on the unit circle. Write an integral formula for the analytic function $f(z)$ in the unit disk such that

$$\lim_{z\to\xi} \Re f(z)=u(\xi)$$

for every ξ on the unit circle.

6. Let f be an analytic function in the unit disk $\mathbb{D}$. Suppose that $|f(z)| \le 1$ in $\mathbb{D}$. Prove that then

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

7. Show that the equation

$$e^z - z = \lambda,$$

where $\lambda > 1$, has exactly one root in the left half-plane $\{\Re z < 0\}$.

8. Let f be a function, analytic and bounded in the strip $\{|\Im z| < \pi/2\}$. Suppose that $f(\ln n)=0$ for all $n \in \mathbb{N}$. Prove that f is identically zero.

9. Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Show that f can be expressed as $f(z)=g(z)/h(z)$ where g and h are entire.

10. Describe the set of all harmonic functions $u(x,y)$ in $\mathbb{C}$ such that the product $(x^2-y^2)u(x,y)$ is harmonic in $\mathbb{C}$.