Complex Analysis
Qualifying Examination

August 2015

1. Find every complex number z for which the infinite series $\sum_{n=1}^{\infty} \left(\frac{2015+i}{2015-i}\right)^{n^2} \left(\frac{z-2015}{z+2015}\right)^n$ converges.

2. Determine every complex number w that can be written in the form $\sin(z)$ for some complex number z having positive imaginary part. In other words, what is the image of the open upper half-plane under the sine function?

3. Prove that $\displaystyle\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}$.

4. When n is an integer, the Bessel function $J_n(z)$ can be defined to be the coefficient of t^n in the Laurent series about the origin of
$$\exp\left(\frac{1}{2}z\left(t-\frac{1}{t}\right)\right)$$
(series with respect to the variable t). Use this definition to show that $J_{-n}(z) = (-1)^n J_n(z)$.

5. When the variable z is restricted to the first quadrant (where $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$), how many zeroes does the polynomial $z^{2015} + 8z^{12} + 1$ have?

6. Suppose f is an entire function such that $f(x+0i)$ is real for every real number x, and $f(0+yi)$ is real for every real number y. Prove the existence of an entire function g such that $f(z) = g(z^2)$ for every complex number z.

7. Does there exist a holomorphic function that maps the open unit disk surjectively (but not injectively) onto the whole complex plane?

8. Determine the group of holomorphic bijections (automorphisms) of $\{\, z \in \mathbb{C} : |z| > 1 \,\}$, the complement of the closed unit disk.

9. On the punctured plane $\mathbb{C} \setminus \{0\}$, can the function $e^{1/z}$ be obtained as the pointwise limit of a sequence of polynomials in z?

10. Prove that if f_1 and f_2 are holomorphic functions with no common zero in a region of the complex plane, then there exist holomorphic functions g_1 and g_2 such that $f_1 g_1 + f_2 g_2$ is identically equal to 1 in the region.
    [Exactly 100 years ago, the algebraist J. H. M. Wedderburn proved this proposition by applying Mittag-Leffler's theorem.]