Complex Analysis
Qualifying Examination

7 August 2019

1. State Picard's great theorem, the Riemann mapping theorem, and the Schwarz lemma.

2. Find necessary and sufficient conditions on a pair of complex numbers α and β for validity of the following property:

$$|\alpha z + \beta \bar{z}| = |z| \qquad \text{for all values of the complex variable } z.$$

3. The series $\sum_{n=1}^{\infty} \left(\frac{2z}{z+1} \right)^n$ converges in some neighborhood of 0 to a function that admits an analytic continuation $f(z)$ to a neighborhood of the point -1. Determine the value $f(-1)$.

4. How many zeros does the polynomial $z^{2019} + 8z + 7$ have in the disk where $|z| < 1$?

5. The analytic functions on the unit disk form a commutative ring under the operations of addition and multiplication. Prove that this ring is an integral domain. In other words, prove that if f is not the zero function and g is not the zero function, then the product fg is not the zero function.

6. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\log(1+x^2)}{4+x^2}\, dx = \pi \log(3)$.

7. Suppose f is a bounded holomorphic function on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane), and f is periodic with period 1, that is, $f(z+1) = f(z)$ when $\operatorname{Re}(z) > 0$. Prove that f must be a constant function.

8. Suppose a metric d is defined on the space of entire functions as follows:

$$d(f,g) = \sum_{n=1}^{\infty} \min \left\{ \frac{1}{2^n}, \max_{|z| \le n} |f(z) - g(z)| \right\}.$$

   Is the operator of differentiation (the operator sending f to f') continuous on this metric space of functions? Explain why or why not.

9. Suppose f is an entire function having the property that $(\operatorname{Re} f(z))^2 \le (\operatorname{Re} f(0))^2$ when $|z| < 1$. Prove that f must be a constant function.

10. Determine the most general entire function f having the property that $|f(z)| = 1$ when $\operatorname{Im}(z) = 0$.