Complex Analysis
Qualifying Examination

August 2022

1. Determine the maximum of the absolute value of the expression $z^{20} - z^{22}$ when the absolute value of the complex variable z is less than or equal to 2.

2. Suppose $\sum_{n=1}^{\infty} c_n z^n$ is a power series in which for every positive integer n, the complex number c_n has the property that $n^{20} \leq |c_n| \leq n^{22}$. What can you say about the radius of convergence of the power series?

3. Prove that $\displaystyle\int_0^{\infty} \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi}{\sqrt{2}}$.

4. Find a linear fractional transformation T (in other words, a Möbius transformation) such that $T(0) = 2$, $T(2) = 0$, and $T(20) = 22$.

5. If the function $\dfrac{1}{z^{20} - \sin(z^{22})}$ is expanded in a Laurent series $\sum_{n=-20}^{\infty} c_n z^n$ converging in a punctured neighborhood of the origin, what is the value of the coefficient c_2?

6. Determine every entire function f having the property that $|f(z)| \leq |\sin(z)|$ for all values of the complex variable z.

7. Consider the family of power series $\sum_{n=0}^{\infty} c_n z^n$ having radius of convergence equal to 22. Must every sequence of holomorphic functions in this family have a subsequence that converges uniformly on the smaller disk $\{ z \in \mathbb{C} : |z| < 20 \}$ to either a holomorphic function or infinity? Explain.

8. Suppose f is an entire function that maps the real axis into the real axis and maps the imaginary axis into the imaginary axis. Prove that the function f must be odd. In other words, $f(-z) = -f(z)$ for every complex number z.

9. Prove that infinitely many values of the complex variable z exist for which $\sin(z) = z$.

10. Prove that the punctured disk $\{ z \in \mathbb{C} : 0 < |z| < 2 \}$ cannot be mapped onto the annulus $\{ z \in \mathbb{C} : 20 < |z| < 22 \}$ by a bijective holomorphic function.