Complex Analysis Qualifying Exam, August 2025

$\mathbb{D}$ denotes the unit disc $B(0,1)$. For an open set $G \subseteq \mathbb{C}$, $H(G)$ denotes the space of analytic functions on G.

Problem 1: Find the Laurent series expansion of $f(z) = \frac{4z-z^2}{(z^2-4)(z+1)}$ in the region $\{0 < |z+1| < 1\}$.

Problem 2: Let U, V be simply connected open subsets of $\mathbb{C}$ with $U \cap V \neq \emptyset$. Prove that every component of $U \cap V$ is also simply connected.

Problem 3: Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \ \cdots \ (2n-1)}{2 \cdot 4 \cdot 6 \ \cdots \ 2n}\, \pi \ .$$

Problem 4: Let C be a circle in $\mathbb{C}$, α, $\beta \in \mathbb{C} \setminus C$. Show that there is a Möbius transformation T with $T(C) = C$ and $T(\alpha) = \beta$.

Problem 5: Show that there is a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ such that
(i) $p_n(z) \to 0$, uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$,
(ii) $p_n(z) \to 1$, uniformly on compact subsets of $\mathbb{R} \setminus \{0\}$,
and
(iii) $p_n(0) \to \infty$.

Problem 6: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that for any $a \in \mathbb{R}$, at least one coefficient in the Taylor expansion of f about a is a rational number. Prove that f is a polynomial.

Problem 7: Suppose f and g are analytic in a region containing $\overline{\mathbb{D}}$. Suppose further that f has a simple zero at $z = 0$ and vanishes nowhere else on $\overline{\mathbb{D}}$. Set $f_\varepsilon(z) = f(z) + \varepsilon g(z)$. Show that if ε is sufficiently small, then
(i)$f_\varepsilon(z)$ has a unique zero in $\overline{\mathbb{D}}$,
and
(ii) if z_ε is this zero, then the mapping $\varepsilon \to z_\varepsilon$ is continuous.

Problem 8: Let $G \subseteq \mathbb{C}$ be an open set, with an exhaustion $\{K_n\}_{n=1}^{\infty}$ by compact sets satisfying $K_n \subseteq int(K_{n+1})$. Show that for a set $\mathcal{F} \subseteq H(G)$ the following are equivalent:
(i) $\mathcal{F}$ is normal,
(ii) for every $\varepsilon > 0$ there is $c > 0$ such that $\{cf \,|\, f \in \mathcal{F}\} \subseteq B(0,\varepsilon)$, where $B(0,\varepsilon) = \{f \in H(G) \,|\, \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\max_{z \in K_n} |f(z)|}{1+\max_{z \in K_n} |f(z)|} < \varepsilon\}$.

Problem 9: Show that the Hadamard factorization of $f(z) = e^z - 1$ equals
$e^{z/2} z \Pi_{n=1}^{\infty} (1 + \frac{z^2}{4n^2\pi^2})$.
Hint: Consider the function $g(z) = \frac{e^z-1}{ze^{z/2}}$ $(g(0) = 1)$.

Problem 10: *Sketch* a proof of why a continuous function on an open set in $\mathbb{C}$ that has the mean value property must be harmonic.