Complex Analysis Qualifying Examination

January 12, 2011

1. Suppose $f(z) = \tan(\omega_1 z)\tan(\omega_2 z)\tan(\omega_3 z)$, where ω_1, ω_2, and ω_3 denote the three cube roots of 1. Show that the nth coefficient in the Maclaurin series of f is equal to 0 when n is not a multiple of 3.

2. The *Schwarzian derivative* of f is the expression $\left(\dfrac{f''}{f'}\right)' - \dfrac{1}{2}\left(\dfrac{f''}{f'}\right)^2$. Show that if f is a linear fractional transformation (Möbius transformation), then the Schwarzian derivative of f is identically equal to 0.

3. Suppose that f is an entire function such that $|f(z)| > 1/(1 + |z|)$ for all z. Prove that f must be a constant function.

4. Suppose Ω is a connected open subset of $\mathbb{C}$, and $u\colon \Omega \to \mathbb{R}$ is a nonconstant harmonic function. Prove that $u(\Omega)$, the image of u, is an open subset of $\mathbb{R}$.

5. Use the residue theorem to prove that $\displaystyle\int_0^\infty \frac{1}{1+x^{2011}}\,dx = \frac{\pi/2011}{\sin(\pi/2011)}$.

6. Suppose f is a holomorphic function (not necessarily bounded) on $\{\, z \in \mathbb{C} : |z| < 1 \,\}$, the open unit disk, such that $f(0) = 0$. Prove that the infinite series $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact subsets of the open unit disk.

7. Either give an example of a holomorphic function (not necessarily one-to-one) that maps the punctured unit disk $\{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$ surjectively onto the unit disk $\{\, z \in \mathbb{C} : |z| < 1 \,\}$ or prove that no such function exists.

8. Suppose $f(z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ when $|z| < 1$. Show that this f (called the *dilogarithm function*) admits an analytic continuation to $\mathbb{C} \setminus [1, \infty)$, the complex plane with a slit along the positive real axis from 1 to ∞.

9. An *exponential polynomial* is a function of the form $\sum_{k=1}^{n} p_k(z)e^{a_k z}$, where each p_k is a polynomial (not identically equal to 0), and $a_1, \ldots, a_n$ are distinct complex numbers. Characterize the exponential polynomials that have no zeroes.

10. State and prove *one* of the following theorems: the Riemann mapping theorem, Rouché's theorem, or Runge's theorem about approximating holomorphic functions by polynomials.