COMPLEX ANALYSIS QUALIFYING EXAM
JANUARY 2021.

1. Let R denote the region between the circles $|z| = 1$ and $|z - 1/2| = 1/2$. Find a conformal map from R to the open unit disk $|w| < 1$.

2. Let f be an analytic function on the upper half-plane which maps the line $\text{Im}(z) = 1$ to $\mathbb{R}$ and satisfies the estimate $|f(z)| < \log(1 + |z|)$. Show that f is constant.

3. State the maximum modulus principle and Riemann's removable singularity theorem. State and prove the Schwarz lemma.

4. Show that all of the zeros of the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ are contained in the disk $|z| < 1 + \max_j |a_j|$. (Cauchy, 1829).

5. Let f be analytic on the complex plane except for a finite set of poles S such that $S \cap \mathbb{Z} = \emptyset$. A standard procedure to evaluate $\sum_{n=-\infty}^{\infty} f(n)$ is to consider the sequence of integrals $\int_{C(n)} \pi \cot(\pi z) f(z)\, dz$ where $C(n)$ is the square with vertices $(n + 1/2)(\pm 1 \pm i)$. Use this process to evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$, justifying all steps.

6. Show that $\Pi_{n=1}^{\infty} (1 - e^{-n^2} e^{zn})$ defines an entire function f. Denote by $\{a_k\}$ the zeros of f, listed with multiplicity, and by E_p the standard elementary factors. Find the smallest integer p such that the product $\Pi_{n=1}^{\infty} E_p(\frac{z}{a_n})$ converges.

7. Prove that if E is a compact, connected subset of $\mathbb{C}_\infty$ (the Riemann sphere) that contains more than one point, then each connected component of $\mathbb{C}_\infty - E$ is biholomorphic to the unit disk.

8. Let f and g be entire functions such that $f^4 - g^4 = 1$. Show that f and g are constant.

9. Show that the series $L_2(z) = \sum_{n=1}^{\infty} z^n/n^2$ can not be analytically continued to an open set containing the closed disk $\{z \mid |z| \leq 1\}$. Show that $L_2(z)$ has an analytic continuation to $\mathbb{C} - [1, \infty)$ which satisfies the functional equation
$$L_2(1 - z) = -L_2(z) - \log(z)\log(1 - z) + \pi^2/6$$
which is valid on the complement of $(-\infty, 0] \cup [1, \infty)$.

10. If $f_k(z) = \sin(kz)/k$, is the family $\{f_k\}_{k=1}^{\infty}$ a normal family in the unit disk? Explain why or why not.