Qualifying Examination in Real Variables, August 2014

General Instructions:

(1) Use a separate sheet of paper for each problem.

(2) Unless stated otherwise, you may use results from Folland's book, but you need to state them carefully (it is not necessary to remember their names).

Problems:

(1) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be continuous, and assume that for every $x \in [0, 1]$ the sequence $(f_n(x))$ is decreasing. Suppose that f_n converges pointwise to a continuous function f.
Show that this convergence is uniform.

(2) Let $f \in L^1(0, \infty)$. For $x > 0$, define

$$g(x) = \int_0^\infty f(t)e^{-tx}dt.$$

Prove that $g(x)$ is differentiable for $x > 0$ with derivative

$$g'(x) = \int_0^\infty -tf(t)e^{-tx}dt.$$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that

$$\int_a^b f(x)dx = 0 \text{ for every } a < b.$$

Show that $f(x) = 0$ for almost every $x \in \mathbb{R}$.

(4) Let f be Lebesgue measurable on $[0, 1]$ with $f(x) > 0$ a.e. Suppose (E_k) is a sequence of measurable sets in $[0, 1]$ with the property that $\int_{E_k} f(x)dx \to 0$ as $k \to \infty$.
Prove that $m(E_k) \to 0$ as $k \to \infty$.

(5) Let (f_n) be a sequence of continuous functions on $[0, 1]$ such that for each $x \in [0, 1]$ there is an $N = N_x$ so that

$$f_n(x) \geq 0 \text{ for all } n \geq N_x.$$

Show that there is an open nonempty set $U \subset [0, 1]$ and an $N \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq N$ and all $x \in U$.

(6) a) Define the w^*-topology on the dual X^* of a Banach space X.

b) Let X be an infinite dimensional Banach space. What is the w^*-closure of

$$S_{X^*} = \{x^* \in X^* : \|x^*\| = 1\}?$$

(As usual, prove your answer.)

(7) a) State the Riesz Representation Theorem for the dual $L_p^*(\mu)$ of $L_p(\mu)$, $1 \le p < \infty$.

b) Let μ be a finite measure on the measurable space (Ω, Σ). Prove the following part of the proof of the above Theorem: If $F \in L_p^*(\mu)$, then there exists an $h \in L_1(\mu)$ so that

$$F(\chi_A) = \int_A h \, d\mu \text{ for all } A \in \Sigma.$$

(8) Assume that (x_n) is a weakly converging sequence in a Hilbert space H. Show that there is a subsequence (y_n) of (x_n) so that

$$\frac{1}{n} \sum_{j=1}^{n} y_j$$

converges in norm.

(9) Show that a linear functional ϕ on a Banach space X is continuous if and only if $\{x \in X : \phi(2x) = 3\}$ is norm closed.

(10) Let $C^1[0,1]$ be the space of functions $f \in C[0,1]$ such that f' exists and is continuous in $[0,1]$. The space $C^1[0,1]$ is given the supremum norm. Define $T : C^1[0,1] \to C[0,1]$ by $Tf = f'$ for $f \in C^1[0,1]$. Show that T has a closed graph and that T is not bounded. Decide if $C^1[0,1]$ (together with the supremum norm) is a Banach space or not. (Explain your answer).