Real Analysis Qualifying Exam, August, 2016

1. Let $\mathcal{A}$ be the set of all real valued functions on $[0,1]$ for which $f(0)=0$ and $|f(t)-f(s)|^{1/2} \leq t-s$ for all $0 \leq s < t \leq 1$.

a. Prove that $\mathcal{A}$ is a compact subset of $C[0,1]$.

b. Prove that $\mathcal{A}$ is a compact subset of $L_1[0,1]$

Don't forget to check that $\mathcal{A}$ is closed in the given Banach spaces.

2. (a) Let $f(x)$ be a real valued function on the real line that is differentiable almost everywhere. Prove that $f'(x)$ is a Lebesgue measurable function.

(b) Prove that if f is a continuous real valued function on the real line, then the set of points at which f is differentiable is measurable.

3. (a) Let f be a real valued function on the unit interval $[0,1]$. Prove that the set of points at which f is discontinuous is a countable union of closed subsets.

(b) Prove that there does not exist a real valued function on $[0,1]$ that is continuous at all rational points but discontinuous at all irrational points.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let (f_n) be a sequence of measurable functions on X that converges pointwise to zero. Prove that (f_n) converges in measure to zero. Show that the converse is false for $[0,1]$ with Lebesgue measure.

5. If f is Lebesgue integrable on the real line, prove that $\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h)-f(x)|dx = 0$.

6. Prove or disprove that there exists a sequence (P_n) of polynomials such that $(P_n(t))$ converges to one for every $t \in [0,1]$ but $\int_0^1 P_n(t)\,dt$ converges to two as $n \to \infty$.

7. Let (f_n) be a uniformly bounded sequence of continuous functions on $[0,1]$ that converges pointwise to zero. Prove that 0 is in the norm closure in $C[0,1]$ of the convex hull of (f_n) (the norm is of course the sup norm on $C[0,1]$).

8. Assume that X is a reflexive Banach space and ϕ is a continuous linear functional on X. Prove that ϕ achieves its norm; that is, prove that there is a norm one vector x in X such that $\phi(x) = \|x\|$. Show by example that there is a continuous linear functional on the Banach space ℓ_1 that does not achieve its norm.

9. Suppose that X is a non separable Banach space. Prove that there is an uncountable subset A of the unit ball of X such that for all $x \neq y$ in X, $\|x-y\| > .9$.

10. If A is a Borel subset of the line, then $E = \{(x,y) : x-y \in A\}$ is a Borel subset of the plane. If the Lebesgue measure of A is 0, then the Lebesgue measure of E is 0.