Real Analysis Qualifying Exam, August, 2017

(1) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on X. Prove, directly from the definition of convergence almost everywhere, that if $\sum_n \mu[|f_n| > 1/n] < \infty$, then the sequence $\{f_n\}$ converges almost everywhere to zero. Deduce that every sequence of measurable functions that converges in measure to zero has a subsequence that converges almost everywhere to zero.

(2) Show that there is a sequence of nonnegative functions $\{f_n\}$ in $L^1(\mathbb{R})$ such that $\|f_n\|_{L^1(\mathbb{R})} \to 0$, but for any $x \in \mathbb{R}$, $\limsup_n f_n(x) = \infty$.

(3) Construct a sequence of nonnegative Lebesgue measurable functions $\{f_n\}$ on $[0,1]$ such that
(a) $f_n \to 0$ almost everywhere, and
(b) for any interval $[a,b] \subseteq [0,1]$,

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = (b-a).$$

(4) In this problem the measure is Lebesgue measure on $[0,1]$. The norm on $L^\infty[0,1]$ is the essential supremum norm, which for a continuous function is the same as the supremum norm.
(a) Prove or disprove that $L^\infty[0,1]$ is separable in the norm topology.
(b) Recall that $L^\infty[0,1] = (L^1[0,1])^*$, What is the weak* closure in $L^\infty[0,1]$ of the unit ball of $C[0,1]$? Prove your assertion.

(5) Prove that if $a_1, a_2, \ldots, a_N$ are complex numbers, then
(a) $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi i k t)|^p \, dt \le \sum_{k=1}^N |a_k|^p$, if $1 \le p \le 2$, and
(b) $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi i k t)|^p \, dt \ge \sum_{k=1}^N |a_k|^p$, if $2 \le p < \infty$.

(6) Prove that if X is an infinite dimensional Banach space and X^* is separable in the norm topology, then there is a sequence $\{x_n\}$ of norm one vectors in X such that $\{x_n\}$ converges weakly to zero.

(7) Prove or disprove each of the following statements.

(a) "If $\{f_n\}$ is a sequence in $C[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly."

(b) "If $\{f_n\}$ is a sequence in $L^2[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly." (Lebesgue measure on $[0,1]$.)

(8) Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ that converges pointwise to a real valued function f. Prove that for each $a < b$, the function f is continuous at some point of $[a,b]$. (Hint: Let $E_{n,m,k} = [|f_n - f_m| \leq 1/k]$.)

(9) Let X and Y be compact Hausdorff spaces and let S be the set of all real functions on $X \times Y$ of the form $h(x,y) = f(x)g(y)$ with f in $C(X)$ and g in $C(Y)$. Prove or disprove that the linear span of S is dense in $C(X \times Y)$.

(10) Let X be a Hilbert space and assume that $\{x_n\}$ is a sequence in X that converges weakly to zero. Prove that there is a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequence $\|N^{-1}\sum_{k=1}^{N} y_k\|$ converges to zero. **Caution:** The same statement is NOT true in all Banach spaces; not even in all reflexive Banach spaces.

(11) Let $F \subset C([0,1])$ be a family of continuous functions such that

1. the derivative $f'(t)$ exists for all $t \in (0,1)$ and $f \in F$.
2. $\sup_{f\in F} |f(0)| < \infty$ and $\sup_{f\in F} \sup_{t\in(0,1)} |f'(t)| < \infty$.

Prove that F is precompact in the Banach space $C([0,1])$ equipped with the norm $||f|| = \sup_{t\in[0,1]} |f(t)|$.

(12) Let $\{x_n\}$ be a weakly Cauchy sequence in a normed linear space X. Prove that

(a) x_n is norm bounded in X.

(b) There exists x^{**} in X^{**} such that x_n converges weak* to x^{**}, and $\|x^{**}\| \leq \liminf_n \|x_n\|$.