Real Analysis Qualifying Exam

Texas A&M Mathematics, August 2018

Solve any 10 of the following 12 problems. Start the solution of each problem you attempt on a fresh sheet of paper. Good luck!

1. Let μ and ν be positive measures on the same measureable space with ν finite and absolutely continuous with respect to μ. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.

2. Let μ be a positive measure. Suppose that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(\mu)$. Show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies

$$\forall n \geq 1 \quad \left| \int_E f_n \, d\mu \right| < \epsilon.$$

   You may use without proof the result of Problem #1.

3. Let $f : [0,1] \to [0,\infty)$ be Lebesgue measurable. For $n \in \mathbb{N}$ define

$$g_n = \frac{f^n}{1+f^n}.$$

   (a) Explain why $\int_0^1 g_n(t)dt$ exists and is finite for all n.

   (b) Prove that $\lim_n \int_0^1 g_n(t)dt$ exists and find an expression for it. Make sure to state which major theorems you are using in your proof.

4. Consider $C([0,1])$ endowed with its usual uniform norm. Prove or disprove that there is a bounded linear functional φ on $C([0,1])$ such that for all polynomials p, we have $\varphi(p) = p'(0)$, where p' is the derivative of p.

5. (a) Define the *product topology* on the Cartesian product $\prod_{\alpha \in A} X_\alpha$ of a family of topological spaces $(X_\alpha)_{\alpha \in A}$.

   (b) State Tychonoff's compactness theorem.

   (c) State and prove the Banach-Alaoglu theorem. (Hint: Use Tychonoff's theorem).

6. Let (X, d) be a compact metric space.

   (a) Show that X has a countable, dense set $\{x_n \mid n \in \mathbb{N}\}$.

   (b) Let $f_n : X \to [0,\infty)$ be $f_n(x) = d(x, x_n)$. Show that if $x, y \in X$ and $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$, then $x = y$.

7. Let $K > 0$ and let Lip_K be the set of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq K|x - y|$.

   (a) Prove that
   $$d(f_1, f_2) = \sum_{j=0}^{\infty} 2^{-j} \sup_{x \in [-j,j]} |f_1(x) - f_2(x)|$$
   defines a metric on Lip_K.

   (b) Prove that (Lip_K, d) is a complete metric space.

8. Let X, Y be topological spaces. A map $f : X \to Y$ is said to be *proper* if for every compact subset $K \subseteq Y$, the inverse image $f^{-1}(K)$ is compact.

   (a) Suppose X is a compact space and Y is Hausdorff. Prove that every continuous map $f : X \to Y$ is proper.

   (b) Give an example of a continuous map which is not proper.

   (c) Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a proper continuous map. Prove that f is a *closed* map, i.e. $f(C)$ is closed in $\mathbb{R}^n$ whenever C is a closed subset of $\mathbb{R}^m$.

9. Consider the interval $[-\pi, \pi]$ equipped with Lebesgue measure μ. For $n \in \mathbb{Z}$, consider the functions $f_n \in C([-\pi, \pi])$ given by $f_n(t) = e^{int}$.

   (a) Prove that $\mathrm{span}_{\mathbb{C}}\{f_n : n \in \mathbb{Z}\}$ is dense in the space
   $$\mathcal{A} := \{f \in C([-\pi, \pi]) \mid f(-\pi) = f(\pi)\}$$
   with respect to the uniform norm.

   (b) Show that $\{\frac{f_n}{\sqrt{2\pi}} \mid n \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $L^2([-\pi, \pi], \mu)$.

   (c) Is the following statement true or false?:

   "$\forall f \in \mathcal{A}$, $f = \lim_{N \to \infty} \frac{1}{2\pi} \sum_{n=-N}^{N} \langle f, f_n \rangle f_n$ with respect to the uniform norm."

   Give a brief explanation why or why not.

10. Let $(X, \|\cdot\|)$ be a normed linear space and let $(X^*, \|\cdot\|_{X^*})$ denote its dual Banach space of bounded linear functionals. Recall that $\|\varphi\|_{X^*} = \sup_{\|x\|=1} |\varphi(x)|$ for $\varphi \in X^*$.

    (a) Prove that for each $x \in X$ there exists $\varphi \in X^*$ with $\|\varphi\|_{X^*} = 1$ and $\|x\| = \varphi(x)$.

    (b) Prove that the linear map $\iota : X \to X^{**}$ given by
    $$\iota(x)(\varphi) = \varphi(x); \qquad (x \in X, \varphi \in X^*).$$
    is an isometry.

(c) A Banach space X is called *reflexive* if $\iota(X) = X^{**}$. Prove that the Banach space

$$\ell^1 = \{f : \mathbb{N} \to \mathbb{C} \mid \|f\|_1 = \sum_k |f(k)| < \infty\}$$

is not reflexive.
(Hint: Consider a weak-$*$ cluster point of the sequence $(\iota(f_n))_{n\in\mathbb{N}} \subset (\ell^1)^{**}$, where $f_n \in \ell^1$ is the unit vector

$$f_n(k) = \begin{cases} \frac{1}{n}, & k \leq n \\ 0, & k > n \end{cases}$$

11. Let $(g_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ be a sequence of non-negative continuous functions. Assume that for each $k = 0, 1, 2, \ldots$, the limit

$$\lim_{n\to\infty} \int_0^1 x^k g_n(x)dx \qquad \text{exists.}$$

Prove that there exists a unique finite positive Radon measure μ on $[0,1]$ such that

$$\int_0^1 f(x)d\mu(x) = \lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx \qquad \text{for all } f \in C([0,1]).$$

12. Let X be a locally compact Hausdorff space equipped with a Radon probability measure μ. Let $E \subseteq L^2(X,\mu)$ be a closed linear subspace and assume that E is contained in $C_0(X)$. The goal of this problem is to prove that $\dim E < \infty$ by justifying the following steps:

(a) There exists a constant $1 \leq K < \infty$ such that

$$\|f\|_2 \leq \|f\|_u \leq K\|f\|_2 \qquad \text{for all } f \in E,$$

where $\|\cdot\|_u$ denotes the uniform norm. (Hint: Use the closed graph theorem for one of the inequalities.)

(b) For each $x \in X$, there exists a unique $g_x \in E$ such that $\|g_x\|_2 \leq K$ and

$$f(x) = \langle f, g_x \rangle \qquad \text{for all } f \in E.$$

(c) Let $(f_i)_{i\in I}$ be any orthonormal basis for E. Then

$$\sum_{i\in I} |f_i(x)|^2 = \|g_x\|_2^2 \leq K^2 \qquad \text{for all } x \in X.$$

(d) $\dim E = |I| \leq K^2$.