Real Analysis Qualifying Exam

August 2019

Each problem is worth 10 points. Work each problem on a separate piece of paper.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable non-negative function on X. Define $\nu : \mathcal{M} \to [0, \infty]$ by
$$\nu(E) = \int_E f d\mu.$$
   (a) Prove that ν is a measure.
   (b) Prove that $g \in L^1(\nu)$ if and only if $gf \in L^1(\mu)$ and in that case $\int_X g d\nu = \int_X g f d\mu$.

2. (a) State Fatou's lemma.
   (b) State the dominated convergence theorem.
   (c) Let f_n, g_n, h_n, f, g, h be measurable functions on $\mathbb{R}^n$ satisfying $f_n \leq g_n \leq h_n$, $f_n \to f$ a.e., $g_n \to g$ a.e., and $h_n \to h$ a.e. Suppose moreover that $f, h \in L^1$ and $\int f_n \to \int f$, $\int g_n \to g$. Prove that $g \in L^1$ and $\int g_n \to \int g$.

3. Let $\{A_k\}_{k=1}^\infty$ be measurable subsets of a measure space and define B_m to be the set of all points which are contained in at least m of the sets $\{A_k\}_{k=1}^\infty$. Prove that B_m is measurable and
$$\mu(B_m) \leq \frac{1}{m} \sum_{k=1}^\infty \mu(A_k).$$

4. Let E be a subset of $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists an $\eta > 0$ such that for any two Lebesgue measurable sets A, B satisfying $A \subseteq E \subseteq B$ one has $\lambda(B \setminus A) > \eta$, where λ denotes Lebesgue measure.

5. Let $\{A_k\}_{k=1}^\infty$ be Lebesgue measurable sets in $\mathbb{R}^n$ equipped with Lebesgue measure λ.
   (a) Prove that if $A_k \subseteq A_{k+1}$ for all k then $\lambda(\bigcup_{k=1}^\infty A_k) = \lim_{k\to\infty} \lambda(A_k)$.
   (b) Prove that if $A_{k+1} \subseteq A_k$ for all k and $\lambda(A_1) < \infty$, then $\lambda(\bigcap_{k=1}^\infty A_k) = \lim_{k\to\infty} \lambda(A_k)$.
   (c) Give an example showing that without assuming $\lambda(A_1) < \infty$ the conclusion of the previous part does not hold.

6. Let X and Y be Banach spaces. Show that the linear space $X \oplus Y$ is a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$. Also determine (with justification) the dual $(X \oplus Y)^*$.

7. For each $n \in \mathbb{N}$ define on ℓ^∞ the linear functional $\varphi_n(x) = n^{-1} \sum_{k=1}^n x(k)$. Let φ be a weak* cluster point of the sequence $\{\varphi_n\}$. Show that φ does not belong to the image of ℓ^1 under the canonical embedding $\ell^1 \hookrightarrow (\ell^\infty)^*$.

8. Let $T : X \to Y$ be a surjective linear map between Banach spaces and suppose that there is a $\lambda > 0$ such that $\|Tx\| \geq \lambda\|x\|$ for all $x \in X$. Show that T is bounded.

9. Let X be a compact metric space and μ a regular Borel measure on X. Let $f : X \to [0, \infty)$ be a continuous function and for each $n \in \mathbb{N}$ set $f_n(x) = f(x)^{1/n}$ for all $x \in X$. Show that $\int f_n \, d\mu \to \mu(\operatorname{supp} f)$ as $n \to \infty$, where $\operatorname{supp} f = \{x \in X : f(x) > 0\}$.

10. Let X be a compact metric space and let $x \in X$. Suppose that the point mass δ_x is the weak* limit of a sequence of atomless Radon measures on X (viewing all of these measures as elements of $C(X)^*$). Show that every neighborhood of x is uncountable.