Qualifying Examination in Real Variables, August 2020

General Instructions:

(1) Use a separate sheet of paper for each problem.

(2) Unless stated otherwise, you may use results from Folland's book. If you do not remember their names you can state them.

Problems:

(1) Let $f \in L^1(\mathbb{R})$. Stating any theorems that you use, compute

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)|^{1/n}\, dx.$$

(2) Let $f(x)$ be a real-valued continuous function on $[0,1]$ satisfying $f(0) = 0$. Given $\varepsilon > 0$, prove that there is a polynomial $p(x)$ such that

$$\|f(x) - x^{1/2} p(x)\|_\infty < \varepsilon.$$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that

$$\int_a^b f(x)\, dx = 0 \text{ for every } a < b.$$

Show that $f(x) = 0$ for almost every $x \in \mathbb{R}$.

(4) Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t)\, dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx.$$

[Hint: first prove the claim under the assumption that $f(x) \geq 0$.]

(5) Let X be an infinite dimensional Banach space. Show that the weak closure of the sphere $S_X = \{x \in X : \|x\| = 1\}$ is the unit ball $B_X = \{x \in X : \|x\| = 1\}$.

(6) Let (A_k) be a sequence of measurable subsets of a measure space $(X, \mathcal{M}, \mu)$ and let B_m be the set of all $x \in X$ which are contained in at least m of the sets A_k, $k \in \mathbb{N}$.

Prove that B_m is measurable and that

$$\mu(B_m) \leq \frac{1}{m} \sum_{k=1}^{\infty} \mu(A_k).$$

(7) (a) State Tietze's Extension Theorem.

(b) Let $n \in \mathbb{N}$ and let $(x_j)_{j=1}^n \subset [0,1]$ and $(r_j)_{j=1}^n \subset \mathbb{R}$ be given. Show that there is a continuous function $f : [0,1] \to \mathbb{R}$, with the property that $f(x_j) = r_j$, $j = 1, 2 \ldots, n$, and

$$\int_0^1 f(x)\, dx = 0.$$

(8) (a) Show that $C[0,1]$ can be naturally viewed as a subspace of $L^2[0,1]$ (on $[0,1]$ we consider the Lebesgue measure) by proving that each equivalence class in $L^2[0,1]$ contains at most one function in $C(K)$.

(b) Let $T : L^2(\mu) \to L^2(\mu)$ be a bounded linear map satisfying $T(C(K)) \subseteq C(K)$. Show that the map $f \mapsto T(f)$ from $C[0,1]$ to itself is bounded with respect to the supremum norm.

(9) (a) Let $C[0,1]$ be the Banach space of real-valued continuous functions on $[0,1]$. Find the extreme points of the unit ball of $C[0,1]$.

(b) Show that $C[0,1]$ is not isometrically isomorphic to a dual space of a Banach space.

(10) Let μ be a Borel measure on $[0,1]$ with $\mu([0,1]) = 1$.

(a) Show that if μ is atomless, then for any $0 < r < 1$ there is a measurable $A \subset [0,1]$, with $\mu(A) = r$.
Recall that $A \subset [0,1]$ is called an *atom* for μ if $\mu(A) > 0$, and for all measurable $B \subset A$, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

(b) Show that μ is atomless if and only if for each $n \in \mathbb{N}$ there is a partition of $[0,1]$ into n sets, $A_1, A_2, \ldots, A_n$, with $\mu(A_j) = \frac{1}{n}$, for $j = 1, 2, 3, \ldots n$.