REAL ANALYSIS QUALIFYING EXAM

FALL 2025

The ten problems below are equally weighted. Please start the solution of each problem you attempt on a new sheet. Make sure to properly mention any named theorem that you need in any of your solutions.

In these problems, m, dx, and dy all denote the Lebesgue measure.

(1) Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.

(2) Let $\mathcal{L}$ be the σ-algebra of all Lebesgue measurable subsets of $[0,1]$, μ be a σ-finite measure on $([0,1], \mathcal{L})$, which is absolutely continuous with respect to the Lebesgue measure. Let $f \in L^1([0,1], \mu)$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x) \sin nx \, d\mu(x) = 0.$$

(You may use the Riemann–Lebesgue lemma without proof.)

(3) Let $(X, \mathcal{M}, \mu)$ be a complete measure space. A sequence $\{f_n\}$ of measurable real-valued functions on X is said to converge in measure to a measurable real-valued function f provided that for each $\epsilon > 0$, $\lim_{n\to\infty} \mu\{x \in X : |f_n(x) - f(x)| > \epsilon\} = 0$. Assume $\mu(X) < \infty$. Show that $\{f_n\} \to f$ in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f.

(4) Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Recall that a sequence $\{f_n\}_n \subset L^2(X,\mu)$ is said converge to $f \in L^2(X,\mu)$ weakly if for any $g \in L^2(X,\mu)$,

$$\lim_{n\to\infty} \int_X f_n(x) g(x) d\mu = \int_X f(x) g(x) d\mu.$$

In this case, prove that there is a subsequence $\{f_{n_k}\}$ such that $\left\{\frac{f_{n_1}+f_{n_2}+\cdots+f_{n_k}}{k}\right\}_k$ converges to f strongly.

(5) Let A be a subset of $\mathbb{R}$. Recall that for two subsets A and B of the vector space $\mathbb{R}$, $A + B := \{x | x = a + b,\, a \in A,\, b \in B\}$. Suppose that $A \subseteq \mathbb{R}$ is Lebesgue measurable and $A + A \subseteq A$. Prove that if $m(\mathbb{R} \setminus A) < \infty$ then $A = \mathbb{R}$.

(6) Let $1 < p < \infty$ and suppose $f_n \in L^p([0,1])$, $\|f_n\|_p \leq 1$, and $f_n(x) \to 0$ for almost every x. Show that $f_n \to 0$ weakly in $L^p([0,1])$.

(7) Let W be any vector space, and suppose that $u, v_1, \ldots, v_k$ are linear functionals on W. Endow W with the weakest topology so that the functionals $v_1, \ldots, v_k$ are continuous. Suppose that u is also continuous in this topology. Prove that u is a linear combination of the v_j.

(8) Suppose X and Y are Banach spaces, and A_n, $n = 1, 2, 3, \ldots$ are bounded linear operators from X to Y. Suppose also that $A_n \to A$ in the weak operator topology, i.e., the weakest topology on the space of bounded linear operators $\mathcal{L}(X, Y)$ in which the maps

$$E_{\ell,x} : \mathcal{L}(X, Y) \ni A \mapsto \ell(Ax), \quad x \in X, \ \ell \in Y^*,$$

are continuous. Show $\sup_n \|A_n\| < \infty$.

(9) Let X be a Banach space for which X^* is separable. Prove that X is separable.

(10) Let $V \subset C([0, 1])$ denote the linear span of the polynomials $\{x^{3n} : n = 0, 1, 2, \ldots\}$. For which values of p, $1 \leq p \leq \infty$, is V dense in $L^p([0, 1])$ (equipped with Lebesgue measure on $[0, 1]$). Prove your answer.