Real analysis qualifying exam

January 2012

Each problem is worth ten points. Work each problem on a separate piece of paper.

1. Let A be the subset of $[0,1]$ consisting of numbers whose decimal expansions contain no sevens. Show that A is Lebesgue measurable, and find its measure. Why does non-uniqueness of decimal expansions not cause any problems?

2. Let the functions f_α be defined by

$$f_\alpha(x) = \begin{cases} x^\alpha \cos\frac{1}{x}, & x > 0, \\ 0, & x = 0. \end{cases}$$

Find all values of $\alpha \geq 0$ such that

(a) f_α is continuous.
(b) f_α is of bounded variation on $[0,1]$.
(c) f_α is absolutely continuous on $[0,1]$.

Justify your answers.

3. Let $\mathcal{F}$ denote the family of functions on $[0,1]$ of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx),$$

where a_n are real and $|a_n| \leq 1/n^3$. State a general theorem and use that theorem to prove that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on $[0,1]$.

4. Let H be a Hilbert space and $W \subset H$ a subspace. Show that $H = \overline{W} \oplus W^\perp$, where $\overline{W}$ is the closure of W. **Note.** Do not just state this as a consequence of a standard result: prove the result.

5. Suppose A is a bounded linear operator on a Hilbert space H with the property that

$$\|p(A)\| \leq C \sup\{|p(z)| : z \in \mathbb{C}, |z| = 1\}$$

for all polynomials p with complex coefficients, and a fixed constant C. Show that to each pair $x, y \in H$, there corresponds a complex Borel measure μ on the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\langle A^n x, y\rangle = \int z^n \, d\mu(z), n = 0, 1, 2, \ldots.$$

6. Let ϕ be the linear functional

$$\phi(f) = f(0) - \int_{-1}^{1} f(t)\, dt.$$

(a) Compute the norm of ϕ as a functional on the Banach space $C[-1, 1]$ with uniform norm.
(b) Compute the norm of ϕ as a functional on the normed vector space $LC[-1, 1]$, which is $C[-1, 1]$ with the L^1 norm.

Justify your answers.

7. Let X be a normed space, and $A \subset X$ a subset. Show that A is bounded (as a set) if and only if it is weakly bounded (that is, $f(A) \subset \mathbb{C}$ is bounded for each $f \in X^*$).

8. Let X be a topological vector space.

(a) Define what this means.
(b) Let $A \subset X$ be compact and $B \subset X$ be closed. Show that $A + B \subset X$ is closed.
(c) Give an example indicating that the condition "A closed" is insufficient for the conclusion.

9. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f, f_n \in L^3(X, d\mu)$ for $n \in \mathbb{N}$ be functions such that $f_n \to f$ μ-a.e. and $|f_n| \leq M$ for all n. Let $g \in L^{3/2}(X, d\mu)$. Show that

$$\lim_{n\to\infty} \int f_n g \, d\mu = \int f g \, d\mu.$$

10. Let X be a σ-finite measure space, and $f_n : X \to \mathbb{R}$ a sequence of measurable functions on it. Suppose $f_n \to 0$ in L^2 and L^4.

(a) Does $f_n \to 0$ in L^1? Give a proof or a counterexample.
(b) Does $f_n \to 0$ in L^3? Give a proof or a counterexample.
(c) Does $f_n \to 0$ in L^5? Give a proof or a counterexample.