Real Analysis Qualiftying Exam
January 2016

Do as many problems as you can. Start each problem on a separate sheet of paper. Unless otherwise specified, the measure involved in each problem is the **Lebesgue measure**.

#1. Let E be a measurable subset of $[0,1]$. Suppose there exists $\alpha \in (0,1)$ such that

$$m(E \cap J) \geq \alpha \cdot m(J)$$

for all subintervals J of $[0,1]$. Prove that $m(E) = 1$.

#2. Let $f, g \in L^1([0,1])$. Suppose

$$\int_0^1 x^n f(x)dx = \int_0^1 x^n g(x)dx$$

for all integers $n \geq 0$. Prove that $f(x) = g(x)$ a.e.

#3. Let $f, g \in L^1([0,1])$. Assume for all functions $\varphi \in C^\infty[0,1]$ with $\varphi(0) = \varphi(1)$, we have

$$\int_0^1 f(t)\varphi'(t)dt = -\int_0^1 g(t)\varphi(t)dt.$$

Show that f is absolutely continuous and $f' = g$ a.e.

#4. Let $\{g_n\}$ be a sequence of measurable functions on $[0,1]$ such that

(i) $|g_n(x)| \leq C$, for a.e. $x \in [0,1]$

(ii) and $\lim_{n\to\infty} \int_0^a g_n(x)dx = 0$ for every $a \in [0,1]$.

Prove that for each $f \in L^1([0,1])$, we have

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx = 0.$$

#5. (a) Let X be a normed vector space and Y be a closed linear subspace of X. Assume Y is a proper subspace, that is, $Y \neq X$. Show that, for $\forall 0 < \varepsilon < 1$, there is an element $x \in X$ such that $\|x\| = 1$ and

$$\inf_{y \in Y} \|x - y\| > 1 - \varepsilon$$

(b) Use part (a) to prove that, if X is an infinite dimensional normed vector space, then the unit ball of X is *not* compact.

#6. Let $\{f_k\}$ be a sequence of increasing functions on $[0,1]$. Suppose

$$\sum_{k=1}^{\infty} f_k(x)$$

converges for all $x \in [0,1]$. Denote the limit function by f, that is,

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Prove that

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x), \qquad \text{a.e. } x \in [0,1].$$

#7. Suppose $f, g : [a,b] \to \mathbb{R}$ are both continuous and of bounded variation. Show that the set

$$\{(f(t), g(t)) \in \mathbb{R}^2 : t \in [a,b]\}$$

cannot cover the entire unit square $[0,1] \times [0,1]$.

#8. Prove the following two statements:

(a) suppose f is a measurable function on $[0,1]$, then

$$\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$$

(b) If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

#9. Suppose $\{f_n\}$ is a sequence of functions in $L^2([0,1])$ such that $\|f_n\|_{L^2} \leq 1$. If f is measurable and $f_n \to f$ in measure, then

(a) $f \in L^2([0,1])$;

(b) $f_n \to f$ weakly in L^2;

(c) $f_n \to f$ with respect to norm in L^p for $1 \leq p < 2$.

Hint for part (b): Use Vitali convergence theorem, that is, if $g_n \to g$ pointwise a.e. on $[0,1]$ and $\{g_n\}$ is uniformly integrable, then

$$\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$$

Recall that $\{g_n\}$ being uniformly integrable means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\int_A |g_n| < \varepsilon$$

for all n and all measurable $A \subseteq [0,1]$ with $m(A) < \delta$.

Hint for part (c): use the fact that, if g_n is a sequence in $L^p([0,1])$ that converges pointwise a.e. to $g \in L^p([0,1])$, then g_n converges to g in norm if and only if $\{|g_n|^p\}$ is uniformly integrable.

#10. Suppose E is a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = \frac{99}{100}$. Show that there exists a number $x \in [0,1]$ such that for all $r \in (0,1)$,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

Hint: Use the Hardy-Littlewood maximal inequality:

$$m(\{x \in \mathbb{R} : Mf(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_1$$

for all $f \in L^1(\mathbb{R})$. Here Mf denotes the Hardy-Littlewood Maximal function of f.