Real Analysis Qualifying Exam
January, 2018

Solve as many of these ten problems as you can in four hours. Start the solution of each problem you attempt on a fresh sheet of paper.

#1. Suppose $U_1, U_2, \ldots$ are open subsets of $[0,1]$. In each case, either prove the statement or disprove it.

(a) If $\lambda(\bigcap_{n=1}^{\infty} U_n) = 0$, then for some $n \geq 1$, we have $\lambda(\overline{U}_n) < 1$, where λ is Lebesgue measure and $\overline{U}_n$ is the closure of U_n in the usual topology on $[0,1]$.

(b) If $\bigcap_{n=1}^{\infty} U_n = \emptyset$, then for some $n \geq 1$, the set $[0,1] \backslash U_n$ contains a nonempty open interval.

#2. Let X be a separable compact metric space and show that $C(X)$ is separable.

#3. Let $f : [0,1] \to \mathbf{R}$ be a bounded Lebesgue measurable function such that $\int_0^1 f(t) e^{nt} dt = 0$ for every $n \in \{0, 1, 2, \ldots\}$. Prove that $f(t) = 0$ for almost every $t \in [0,1]$.

#4. (a) Prove that every compact subset of a Hausdorff space is closed.

(b) Let $f : X \to Y$ be a bijective continuous function between topological spaces. Suppose that X is compact and Y is Hausdorff and prove that f is a homeomorphism.

(c) Prove or disprove that if X is a dense subset of a topological space Y and if X is Hausdorff in the relative topology, then Y is also Hausdorff.

#5. Prove that the following limit exists and compute its value:

$$\lim_{n \to \infty} \int_0^n \Big(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \Big) e^{-2x} dx.$$

#6. Let X and Y be Banach spaces (over $\mathbf{C}$).

(a) A linear map $T : X \to Y$ is called *adjointable* if $T^* f \in X^*$ for every $f \in Y^*$. Prove that T is adjointable if and only if $T \in B(X,Y)$.

(b) Suppose a bounded linear functional $\Psi : X^* \to \mathbf{C}$ is weak*-continuous. Show (from the defintions) that there exists $x \in X$ such that $\Psi(\phi) = \phi(x)$.

(c) Let $S \in B(Y^*, X^*)$. Prove that S is weak*-weak*-continuous if and only if $S = T^*$ for some $T \in B(X,Y)$.

#7. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n : [0,1] \to \mathbf{R}$.

(a) What does it mean for $\{f_n \mid n \geq 1\}$ to be equicontinuous?

(b) Suppose that for every n, f_n is differentiable and $|f_n'(t)| \leq 1$ for all t. Prove that $\{f_n \mid n \geq 1\}$ is equicontinuous.

(c) Suppose the hypothesis of (b) holds and assume in addition that $|f_n(0)| \leq 1$ for every $n \geq 1$. Prove that there exist a continuous function $f : [0,1] \to \mathbf{R}$ and a subsequence $(f_{n(k)})_{k=1}^{\infty}$ converging uniformly to f.

(d) Show by example that the limit function f need not be differentiable.

#8. Let H be a complex Hilbert space. Given a non-empty set $E \subseteq H$ and $x \in H$, put $\mathrm{dist}(x, E) = \inf\{\|x - y\| : y \in E\}$ and $E^{\perp} = \{x \in H : \langle x, y \rangle = 0 \quad \forall y \in E\}$.

(a) Let $H_0 \subset H$ be a closed subspace and $x \in H$. Prove that there exists $x_0 \in H_0$ such that $\|x - x_0\| = \mathrm{dist}(x, H_0)$.

(b) With x and x_0 as above, prove that $x - x_0$ is orthogonal to H_0.

(c) Prove that $H = H_0 \oplus H_0^{\perp}$ (the algebraic direct sum).

(d) Let $E \subseteq H$ be non-empty. Prove that $(E^{\perp})^{\perp} = E$ if and only if E is a closed subspace.

#9. Let $\mathcal{V}$ be a vector space over $\mathbf{R}$ or $\mathbf{C}$. Recall that a *Hamel basis* for $\mathcal{V}$ is a linearly independent subset of $\mathcal{V}$ whose linear span equals $\mathcal{V}$.

(a) Let $S \subseteq \mathcal{V}$ and suppose the linear span of S equals $\mathcal{V}$. Show that $\mathcal{V}$ has a Hamel basis that is a subset of S.

(b) Suppose $\mathcal{V}$ has an infinite Hamel basis and show that all Hamel bases of $\mathcal{V}$ have the same cardinality.

#10. Suppose $(X, \mathcal{M}, \rho)$ is a finite measure space and $\mathcal{A} \subseteq \mathcal{M}$ is an algebra of sets with a finitely additive complex measure $\mu : \mathcal{A} \to \mathbf{C}$ such that $|\mu(E)| \leq \rho(E)$ for all $E \in \mathcal{A}$. Show that there exists a complex measure $\nu : \mathcal{M} \to \mathbf{C}$ whose restriction to $\mathcal{A}$ is μ and such that $|\nu(E)| \leq \rho(E)$ for all $E \in \mathcal{M}$. (Hint: you may want to consider the subspace $\mathcal{V} \subseteq L^1(\rho)$ that is spanned by the set of characteristic functions 1_E for $E \in \mathcal{A}$, and a certain linear functional on $\mathcal{V}$.)