Topology/Geometry Qualifying Examination
August 2009

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^n$ denotes Euclidean n-dimensional space.

1. Let X and Y be topological spaces with Y is Hausdorff and let $f, g : X \to Y$ be continuous functions. Prove that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

2. Let $\mathcal{B}$ be the collection of all sets $U_{\epsilon;L_1,\ldots,L_r}(x) \subset \mathbb{R}^2$ corresponding to all $x \in \mathbb{R}^2$, $\epsilon > 0$, and finite collections of lines $L_1, \ldots, L_r$ passing through x, where

$$U_{\epsilon;L_1,\ldots,L_r}(x) = (B_\epsilon(x) - \{L_1, \ldots, L_r\}) \cup \{x\}.$$

(a) Show that $\mathcal{B}$ forms a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2$.

(b) Given any line L in $\mathbb{R}^2$, describe the subspace topology that L inherits from $\mathcal{T}$.

(c) Show that $\mathcal{T}$ is not first countable.

3. Metrizable spaces

(a) State the Urysohn Metrization Theorem. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X is second countable.

(b) Show that the uncountable product of intervals $[0, 1]$ is not first countable, and therefore is not metrizable.

4. Let X and Y be locally compact Hausdorff spaces. Prove that any proper map from X to Y is closed.

5. Let C be a connected component of a compact Hausdorff space X and let U be an open set containing C.

(a) Prove that the components of X coincide with the quasi-components of X.

(b) Prove that there exists an open and closed set V such that $C \subset V \subset U$.

6. Let M be a smooth manifold and $N \subset M$ be an embedded submanifold, both without boundary.

(a) Give the definition of a *smooth n–dimensional manifold* M.

(b) Give the definition of a an *embedded, k–dimensional submanifold* $N \subset M$.

(c) Assume that N is compact and that $f : N \to \mathbb{R}$ is a smooth function. Prove that f can be extended to a smooth function on M.

(d) Prove, by giving a counter example, that such an extension need not exist if N is not compact.

7. Let M and N be smooth n-dimensional manifolds and $f : M \to N$ be a smooth map.

(a) Define the *tangent bundle* TM of M. Prove that the tangent bundle TM is a $2n$–dimensional differentiable manifold.

(b) Define the differential $f_*\colon T_pM \to T_{f(p)}N$ at a given point $p \in M$. Prove that if f is a diffeomorphism between M and N, then f_* is a diffeomorphism between the corresponding tangent bundles.

8. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4\}$ be the 2-sphere of radius 2. Compute the Gauss and mean curvatures of M.

9. Let (w, x, y, z) be the standard Cartesian coordinates on $\mathbb{R}^4$. Consider the 1-forms

$$\alpha := -\mathrm{d}w + 3\mathrm{d}x + \frac{\mathrm{d}y}{1+y^2} \quad \text{and} \quad \beta := y\,\mathrm{d}x - \mathrm{d}y + e^x\,\mathrm{d}z\,.$$

Prove that, given a point $p \in \mathbb{R}^4$, there exists a 2-dimensional submanifold $M \subset \mathbb{R}^4$ containing p with the property that α and β vanish when restricted (pulled-back) to M.

10. Let (s, t) be the standard Cartesian coordinates on $\mathbb{R}^2$. Define a smooth map $F : \mathbb{R}^2 \to \mathbb{R}^4$ by

$$(s, t) \ \mapsto \ (s^2, 1, \cos(t), \arctan(s))\,.$$

Compute the pull-back $F^*(\alpha)$, where α is the 1-form above.

11. Let $x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ be standard Cartesian coordinates on $\mathbb{R}^4$. Given smooth functions $f^1(x), f^2(x), f^3(x), f^4(x)$ on $\mathbb{R}^4$, prove that the $y^i = f^i(x)$, $i = 1, \ldots, 4$, define a coordinate system in a neighborhood of $p \in \mathbb{R}^4$ if and only if $\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3 \wedge \mathrm{d}f^4 \neq 0$ at p.