Notation. Let $\mathbb{R}^n$ denote real n-space. Employ the summation convention: any repeated index appearing as a subscript and superscript is summed over.

Show your work.

1.) Let C be a subset of a topological space X.

(a) Prove that if C is connected, then the closure of C is connected.

(b) Prove or give a counter-example to the following statement: if C is connected, then the interior of C is connected.

2.) Prove that a countable product of separable spaces is separable.

3.) **(a)** Is the set of rational numbers $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) locally compact? Prove your answer.

(b) Prove that if a topological space X is locally compact, Hausdorff, and second countable, then it is metrizable.

4.) Let $f : X \to Y$ be a continuous map between topological spaces X and Y.

(a) Define what it means for f to be a quotient (an identification) map.

(b) Prove that if the map $f : X \to Y$ is open and onto, then f is a quotient map.

(c) Let C be the union of the x-axis and the y-axis of $\mathbb{R}^2$ and define $g : \mathbb{R}^2 \to C$ as follows:

$$g(x, y) = \begin{cases} (x, 0) & \text{if } x \neq 0 \\ (0, y) & \text{if } x = 0 \end{cases}.$$

Does the quotient topology on C induced by g coincide with the subspace topology on C induced from the standard topology of $\mathbb{R}^2$? Prove your answer.

5.) **(a)** Give the definition of a paracompact space.

(b) Using the definition of paracompactness only, prove that $\mathbb{R}^n$ (with the standard topology) is paracompact.

(c) Give an example to show that if X is paracompact, it does not follow that for every open covering of $\mathcal{A}$ of X there is locally finite *subcollection* of $\mathcal{A}$ that covers X.

6.) Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of compact subsets of a Hausdorff space X which is closed with respect to finite intersections. Let $K = \bigcap_{\alpha \in A} K_\alpha$.

(a) Suppose that W is an open subset of X such that $K \subset W$. Prove that $K_\alpha \subset W$ for some $\alpha \in A$.

(b) Prove that if K_α is connected for each $\alpha \in A$, then K is connected.

7.) **(a)** State the definition of a smooth n–dimensional manifold.

(b) Define $F : \mathbb{R}^3 \to \mathbb{R}^1$ by $F(x,y,z) = x\cos(z) + y\sin(z)$. Prove that the level set $F^{-1}(0)$ is a smooth 2–dimensional manifold.

8.) Let $X_1, \ldots, X_m$ be linearly independent vector fields on $\mathbb{R}^n$, $m \le n$. Fix the index ranges

$$\begin{aligned} 1 &\le a, b \le m \\ 1 &\le i, j \le n \\ m+1 &\le s, t \le n\,. \end{aligned}$$

Prove the following.

(a) For every $p \in \mathbb{R}^n$ there exists an open set $U \subset \mathbb{R}^n$ containing p and linearly independent 1-forms $\eta^1, \ldots, \eta^n$ on U with the property that $\eta^i(X_a) = \delta^i_a$. Here δ^i_a is the Kronecker delta.

(b) Prove that $[X_a, X_b] \subset \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m\}$ if and only if there exist 1-forms α^s_t on U such that $\mathrm{d}\eta^s = \alpha^s_t \wedge \eta^t$, for all $m+1 \le s \le n$.

9.) Let $Z = \mathbb{R}^{n+1} \backslash \{0\}$. Define an equivalence relation $\sim$ on U by

$$x \sim y \text{ if and only if there exists } \lambda \neq 0 \text{ such that } y = \lambda\, x.$$

Recall that projective n-space is the manifold $\mathbb{P}^n = Z/\sim$. Given $x = (x^0, \ldots, x^n) \in Z$, let $[x] = [x^0 : \cdots : x^n] \in \mathbb{P}^n$ denote the corresponding equivalence class. Fix $p = [1:0] \in \mathbb{P}^1$ and $q = [1:0:0:0] \in \mathbb{P}^3$.

(a) Describe a coordinate chart (U, φ) about p, and coordinate chart (V, ψ) about q.

(b) Let $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ be the Veronese map $\nu([s:t]) = [s^3 : s^2 t : s t^2 : t^3]$. Give the local coordinate expression of ν with respect to the coordinates (U, φ) and (V, ψ).

(c) Express the push-forward $\nu_* : T_p\mathbb{P}^1 \to T_q\mathbb{P}^3$ in terms of the local coordinates.

(d) Express the pull-back $\nu^* : T^*_q\mathbb{P}^3 \to T^*_p\mathbb{P}^1$ in terms of the local coordinates.

10.) Consider a unit speed curve $C : t \mapsto (\alpha(t)\,,\, 0\,,\, \beta(t))$ in $\mathbb{R}^3$ with $\alpha(t) > 0$. Let S be the surface of revolution obtained by rotating C about the z-axis. The $(\alpha(t), \beta(t))$ for which S is of constant Gauss curvature $K = -1$ are characterized by an ordinary differential equation. Identify that equation.