TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
AUGUST 2014

INSTRUCTIONS:.

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1. *Let X be a compact Hausdorff space.*

(a) *Let $n \geq 1$ and*
$$\{\ f_i : X \to \mathbf{R} \mid i = 1, \dots, n\}$$
be a finite family of continuous functions such that, for each pair of distinct points $x, y \in X$, there exists i, $1 \leq i \leq n$, with $f_i(x) \neq f_i(y)$. Show that X is homeomorphic to a subspace of $\mathbf{R}^n$.

(b) *Let $f : X \to X$ be an injective continuous function. Show that there exists a nonempty closed subset A of X such that $f(A) = A$.*

Problem 2. *Let X be a topological space. Show that the intersection of any two dense open subsets of X is also dense.*

Problem 3. *Let X be a locally compact space and let A be a subset of X such that, for every compact subset K of X, the intersection $A \cap K$ is a closed subset of X. Show that A is a closed subset of X.*

Problem 4. *Consider the equivalence relation $\sim$ on $I = [0, 1]$ given by*
$$x \sim y \iff x = y \text{ or } 1/3 < x, y < 2/3,$$
and the quotient space $X = I/\sim$. Prove or disprove each of the following

(a) *X is Hausdorff.*
(b) *X is connected.*
(c) *X is compact.*

Problem 5. *In $\mathbf{R}^3$, set*

$$X_1 = x_1^2 x_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}, \quad X_2 = 2x_1 \frac{\partial}{\partial x_2}, \quad \omega = x_3 dx_1 \wedge dx_2 + x_2^2 dx_1 \wedge dx_3.$$

(a) *Compute $[X_1, X_2]$.*
(b) *Compute $\omega(X_1, X_2)$.*
(c) *Compute $\omega \wedge (x_2 dx_2)$.*
(d) *Compute $d\omega$.*
(e) *Prove that for any point $p \in \mathbf{R}^3$ there are no neighborhood U and coordinate functions y_1, y_2, y_3 on U such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$.*
(f) *On the set $M := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 \neq 0\}$ define the distribution $D = \operatorname{span}(X_1, X_2)$. Prove that for any point $p \in M$ there exist a neighborhood U, coordinate functions (y^1, y^2, y^3) on U, and vector fields Y_1 and Y_2 on U such that $D = \operatorname{span}(Y_1, Y_2)$ and $Y_i = \frac{\partial}{\partial y_i}$, $i = 1, 2$. Give an example of such vector fields Y_1, Y_2 (in the original coordinates (x_1, x_2, x_3)).*

Problem 6. *Define $f : \mathbf{R}^3 \to \mathbf{R}^2$ by $f(x, y, z) = (x^2 + y^2, yz)$. Let (u, v) denote standard coordinates in $\mathbf{R}^2$.*

(a) *Calculate $f^*(udv + vdu)$.*
(b) *Calculate $f_* \left(\frac{\partial}{\partial y}|_{(10,-5,-1)} \right)$.*
(c) *Find all regular values of f.*
(d) *Find all (a, b) in $\mathbf{R}^2$ such that the set $f^{-1}(a, b)$ is a nonempty embedded submanifold of $\mathbf{R}^3$.*

Problem 7. *Suppose M is a smooth n-dimensional manifold and D is a smooth rank k distribution on M. Recall that a p-form η annihilates D if $\eta(X_1, \ldots, X_p) = 0$ whenever $X_1, \ldots, X_p$ are local sections of D. Let $\omega^1, \ldots, \omega^{n-k}$ be smooth local defining forms for D over an open subset $U \subseteq M$, i.e. $D_q = \operatorname{Ker} \omega^1|_q \cap \ldots \cap \operatorname{Ker} \omega^{n-k}|_q \quad \forall q \in U$. Prove that a smooth p-form η defined on U annihilates D if and only if it can be expressed in the form*

$$\eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i$$

for some smooth $(p-1)$-forms $\beta^1, \ldots, \beta^{n-k}$ on U.

Problem 8. *Assume that for any $p = (x, y) \in \mathbf{R}^2$ the inner product $\langle \cdot, \cdot \rangle_p$ is given as follows: if $v_1, v_2 \in T_p\mathbf{R}^2$, then $\langle v_1, v_2 \rangle = \lambda(p)(v_1 \cdot v_2)$, where $v_1 \cdot v_2$ is the standard inner product in $\mathbf{R}^2$ and $\lambda : \mathbf{R}^2 \to \mathbf{R}$ is a smooth positive function. Prove that the Gaussian curvature K of the corresponding Riemannian metric is given by $K = -\dfrac{1}{2\lambda} \Delta \left(\log(\lambda) \right)$, where Δ is the Laplacian, $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.*