TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2020

- *There are 10 problems. Work on all of them and prove your assertions.*
- *Use a separate sheet for each problem and write only on one side of the paper.*
- *Write your name on the top right corner of each page.*

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1. A space X is said to be *locally metrizable* if for all $x \in X$ there is a neighborhood of x that is metrizable in the subspace topology. Show that a compact Hausdorff space X is metrizable if and only if it is locally metrizable.

2. Let M denote the set of (non-oriented) closed line segments of length 1 in $\mathbb{R}^2$.
   (a) Show that M can be given a metric so that the resulting metric space has also the structure of a smooth manifold.
   (b) What is the dimension of M?
   (c) Is M an orientable manifold? Explain your answer.

3. Let S^1 denote the unit circle and consider $X := (S^1)^\omega = S^1 \times S^1 \times \times \cdots$, the countable product of S^1 with itself in the product topology. Fix a prime number p and let $\mathcal{S}_p \subset X$ denote the subspace $\mathcal{S}_p := \{\mathbf{a} = (a_0, a_1, \ldots) \in X \mid a_0 = 1 \text{ and } a_{j+1}^p = a_j,\ j = 1, 2, \ldots\}$. Answer the following questions:
   (a) Is $\mathcal{S}_p$ discrete? Is it compact?
   (b) Show that the multiplication in S^1 gives $\mathcal{S}_p$ the structure of a *totally disconnected topological group.*
   (c) Define $N_k := \{\mathbf{a} \in \mathcal{S}_p \mid a_0 = a_1 = \cdots a_k = 1\}$. Show that each N_k is an open subgroup of $\mathcal{S}_p$ and that the collection $\mathcal{N} = \{N_j \mid j \geq 0\}$ forms a countable neighborhood basis of the identity $\mathbf{1}$ of $\mathcal{S}_p$.
   (d) Is $\mathcal{S}_p$ *second countable*? Explain.

4. Let X be a regular space and let $\mathcal{C} = \{U_k \mid k \in \mathbb{N}\}$ be a countable open cover of X having the property that each closure $\overline{U_j}$ is a paracompact subspace. Show that X is paracompact.

5. Let M be a smooth manifold and TM its tangent bundle. Prove that TM (viewed as a smooth manifold itself) is orientable.

6. Let S^n be the n-dimensional sphere. Denote the trivial vector bundle $S^n \times \mathbb{R}$ over S^n by $\mathbb{1}$ and denote the tangent bundle of S^n by $T(S^n)$. Prove that $\mathbb{1} \oplus T(S^n)$ is isomorphic to the direct sum of $(n+1)$-copies of $\mathbb{1}$.

7. Let ω be a closed 1-form on a smooth manifold M. Prove that ω is exact if and only if

$$\int_c \omega = 0$$

for every closed curve c in M.

8. Let σ be the following 2-form on $\mathbb{R}^3$:

$$\sigma = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

(a) Let η be the restriction of σ on the unit sphere S^2. Show that

$$\int_{S^2} \eta > 0$$

(b) Let ξ be the 2-form on $\mathbb{R}^3 - \{0\}$ given by

$$\xi = \frac{\sigma}{(x^2 + y^2 + z^2)^k}$$

for some $k \in \mathbb{R}$. For what values of k is ξ a closed form? For what values of k is ξ an exact form?

9. Let N be a submanifold of M. A vector field X on M is said to be tangent to N if $X_p \in T_pN \subset T_pM$ for all $p \in N$. Prove that if X and Y are vector fields on M that are both tangent to N, then $[X, Y]$ is also tangent to N.

10. Let M be a complete Riemannian manifold and $\gamma\colon [0,1] \to M$ a smooth curve with

$$\frac{d\gamma}{dt} \neq 0$$

everywhere. Suppose X is a vector field along the curve γ such that

$$\|X(t)\| = 1 \text{ and } \left\langle X(t), \frac{d\gamma}{dt} \right\rangle = 0$$

for all $t \in [0,1]$. Let $\alpha\colon [0,1] \times (-\infty, \infty) \to M$ be defined by

$$\alpha(t,s) = \exp_{\gamma(t)}(sX(t)).$$

Prove that for every fixed $s_0, t_0 \in \mathbb{R}$, the curve $\beta\colon [0,1] \to M$ given by

$$\beta(t) = \exp_{\gamma(t)}(s_0 X(t))$$

is perpendicular to the geodesic $\alpha_{t_0}(s) = \exp_{\gamma(t_0)}(sX(t_0))$.