TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2025

- There are 10 problems. Work on all of them and prove your assertions.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

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1. Recall that a topological space X is *locally compact* if for any point x there is a compact subspace C of X that contains an open neighborhood of x.

   Let X be a locally compact space.

   (a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact?

   (b) What if f is continuous and open?

   Justify your answer.

2. Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Prove your answer.

3. Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

4. Show that every continuous map from $\mathbb{RP}^2$ to the circle $\mathbb{S}^1$ is null-homotopic. [Hint: The lifting properties might be helpful here.]

5. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a continuous map from the circle to the real numbers. Show that there exists a point x of $\mathbb{S}^1$ such that $f(x) = f(-x)$.

6. Suppose X is a closed orientable manifold of dimension n. Let ω be a differential form of degree $(n-1)$ on X. Show that $d\omega$ is 0 at some point.

7. Let M be an oriented closed surface equipped with a Riemannian metric g. Suppose M is diffeomorphic to the torus T^2.

   (a) What is the Euler characteristic $\chi(M)$ of M?

   (b) Use the Gauss–Bonnet theorem to compute the integral of the Gaussian curvature:

   $$\int_M K \, dA.$$

   (c) What can you say about the *sign* of the Gaussian curvature K for any Riemannian metric on the torus.

8. (a) Determine the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$. Justify you answer.
   (b) Determine the Lie algebra of $U(n) = \{A \in M_n(\mathbb{C}) \mid A^*A = AA^* = I\}$. Justify you answer.

9. Show that there does not exist a nowhere vanishing tangent vector field on $\mathbb{S}^{2n}$.

10. Let $M = \mathbb{R}^2 \backslash \{(0,0)\}$. Show that the first de Rham cohomology group $H^1(M)$ is nontrivial by giving an explicit 1-form that is closed but not exact. Justify your answer.