Topology Qualifying Examination
January 2013

Instructions. Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

Notation. $\mathbb{N}$ denotes the positive integers. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}^n$ denotes Euclidean n-dimensional space.

1. Let X be a metric space. Given a cover $\{U_\alpha\}$ of X by subsets of X, a *Lebesgue number* for the cover is a number $\epsilon > 0$ such that if $A \subset X$ and $\operatorname{diam}(A) < \epsilon$, then A is contained in at least one set U_β of the cover.

(a) Prove that every open cover of a compact metric space X has a Lebesgue number.

(b) Prove that if $f : X \to Y$ is a continuous map from a compact space X to a metric space Y, then f is uniformly continuous.

2. Let X and Y be topological spaces. Let $f : X \to Y$ be a quotient map. Define *quotient map*. Show that if Y is connected and $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.

3. Define *paracompact space*. Prove that if X is paracompact, then X is normal.

4. Let X and Y be topological spaces. Let $f : X \to Y$ be a surjective function satisfying the condition that $\operatorname{int}(f(A)) \subset f(\operatorname{int}(A))$ for any subset $A \subset X$. Show that f is continuous.

5. For every $S \subset \mathbb{N}$, let $X_S = \{0, 1\}$ with the discrete topology, and let $X = \Pi_S X_S$ with the product topology. Let $f_n(S)$ be 0 if $n \in S$, and 1 if $n \notin S$. Prove that the sequence $\{f_n\}$ in X does not have a convergent subsequence.

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = (x^2 - y^3, xy, (z - 1)^4)$. For which points $p = (x, y, z)$ is F a diffeomorphism in a neighborhood of p?

7. Consider the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$. Compute the tangent space to S at $p = (1, 0, 1)$ and determine the geodesic going from p to $q = (0, 0, 0)$ as a parameterized curve.

8. Define the cotangent bundle of a differentiable manifold. (Hint: first define the cotangent space at a point.)

9. Describe all smooth surfaces in $\mathbb{R}^3$ with coordinates (x, y, z) such that the pullback of the one-form $\theta := dy - z dx$ is identically zero.

10. Let $r > 0$ be a constant and consider the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid r = x^2 + y^2\}$. Compute the Gauss and mean curvature functions on S. What is the group of isometries of S?

