Geometry-Topology Qualifying Examination January 2009

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Similarly $\mathbb{C}$ denotes the complex numbers, and $\mathbb{C}^n$ denotes complex n-dimensional space.

1. Let M^m be a smooth m-dimensional manifold and let N^n be a closed embedded n-dimensional submanifold of M. Define the tangent bundle $T(M)$ as a $2m$-dimensional smooth manifold. Show that $T(N)$ is a closed embedded submanifold of $T(M)$.

2. If X is countably compact and Y is Hausdorff and second countable, then a continuous bijection $f : X \to Y$ is a homeomorphism. (Note: X is *countably compact* if every countable cover has a finite subcover.)

3. Denote $I = [0, 1]$. Let the space X be the set $I \times I$ with the lexicographic order topology ($(a, b) < (c, d)$ if either $a < c$, or $a = c$ and $b < d$). Prove that X is first countable and compact, but not separable.

4. Let X be a paracompact Hausdorff space. Show that if X contains a dense, Lindelöf subspace S, then X is also Lindelöf.

5. Let G be a topological group and let H be a subgroup of G, and denote by G/H the set of left cosets of H in G. Show that $\pi : G \to G/H$ is an open map.

6. Prove that $M := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^4 - z^3 = 1\}$ is a submanifold of $\mathbb{R}^3$.

7. Classify the minimal surfaces $S \subset \mathbb{R}^3$ with zero Gauss curvature.

8. Let M be a manifold, and $N \subset M$ a submanifold. Suppose that X and Y are smooth vector fields on M with the property that $X_p, Y_p \in T_pN$ for all $p \in N$. Prove that $[X, Y]_p \in T_pN$.

9. Let X and Y be two vector fields on $\mathbb{R}^3$ with the property that X_p and Y_p are linearly independent for all $p \in \mathbb{R}^3$. Pick a 1-form ω on $\mathbb{R}^3$ with the property that $\omega(X) = 0 = \omega(Y)$. Prove that though every point $p \in \mathbb{R}^3$ there exists a surface S such that the tangent spaces T_qS, $q \in S$, are spanned by X_q and Y_q if and only if $\omega \wedge \mathrm{d}\omega = 0$.

10. Prove that the saddle surface $z = xy$ is ruled. Compute its Gauss curvature.

