Zeros of the Eisenstein Series

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Basic Definitions

- Throughout the presentation, $z = x + iy$.
- $\mathbb{H} = \{z \in \mathbb{C} : y > 0\}$
- $\mathbb{D} = \{z \in \mathbb{H} : |z| \geq 1 \text{ and } -\frac{1}{2} \leq x \leq \frac{1}{2}\}$
- $\mathbb{G} = \{z \in \mathbb{H} : -\frac{1}{2} \leq x \leq \frac{1}{2}\}$
- $SL_2(\mathbb{Z})$ is the group of matrices where $\gamma \in SL_2(\mathbb{Z}), \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$
- If $\gamma \in SL_2(\mathbb{Z}) : \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\gamma(z) = \frac{az+b}{cz+d}$.

Introduction to $E_{2k}(z)$

This presentation will deal primarily with the **Eisenstein Series of Weight 2k**. These are the functions which have the Fourier expansions:

$$E_{2k}(z) = 1 + \gamma_{2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z},$$

where

$$\gamma_{2k} = (-1)^k \frac{4k}{B_k},$$

B_k is the k-th Bernoulli number, and $\sigma_{2k-1}(n) = \sum_{a|n} a^{2k-1}$

When $k \geq 2$, $E_{2k}(z)$ is a **Modular Form** for $SL_2(\mathbb{Z})$.

Modular Forms

To be a modular form for $SL_2(\mathbb{Z})$, $E_{2k}(z)$ must be holomorphic on $\mathbb{H}$, including ∞, and satisfy the relations

$$(cz+d)^{2k} f(z) = f(\frac{az+b}{cz+d}) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

- F.K.C Rankin and Swinnerton-Dyer (1970)

Quasimodular Forms

When k=1, $E_2(z)$ is known as a quasimodular form, which fufills the following relation:

$$E_2(\frac{az+b}{cz+d}) = (cz+d)^2 E_2(z) - \frac{6}{\pi} ic(cz+d).$$

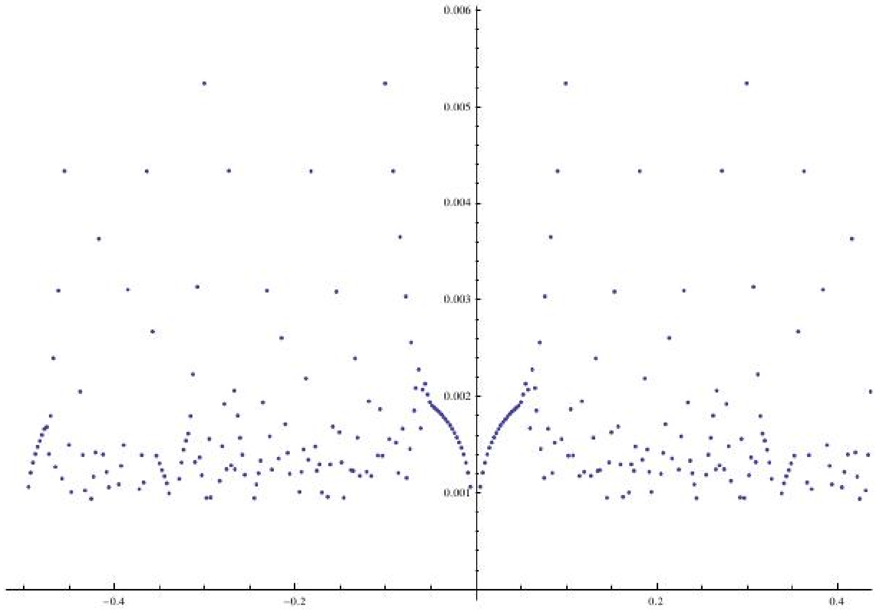
The Fourier expansion of $E_2(z)$ is given by

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}.$$

Zeros of $E_2(z)$

- Basraoui and Sebbar (2012)
- $E_2(z) = 0$ doesn't have any solutions within $\mathbb{D}$.
- $E_2(z) = 0$ has infinitely many zeros in $\mathbb{G}$
- Not much else is known about their general distribution or location.

Graph of $E_2(z) = 0$



Numerical Output of $E_2(z) = 0$

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And this pattern continues

$$0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{4}, -\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{2}{5},$$

$$-\frac{1}{6}, \frac{1}{6}, -\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8},$$

$$-\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, -\frac{3}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \ldots\ldots$$

- Now we switch our focus from $E_2(z)$ to $h(z)$.

$$h(z) = z + \frac{6}{\pi i E_2(z)}$$

The function $h(z)$ is equivariant, which means that for

$$h(\gamma z) = \gamma h(z) \text{ for } \gamma \in SL_2(\mathbb{Z}).$$

Theorem 1

If $E_2(z_0) = 0$ then $h(\gamma z_0) = \frac{a}{c}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
Conversely, if $h(\tau_0) = \frac{a}{c}$ with coprime a, c, then $E_2(\gamma^{-1}\tau_0) = 0$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Consider the case when $E_2(z_0) = 0$ (so $h(z_0) = \infty$), and let $z = \gamma z_0$. Note that $\gamma\infty = \frac{a}{c}$.

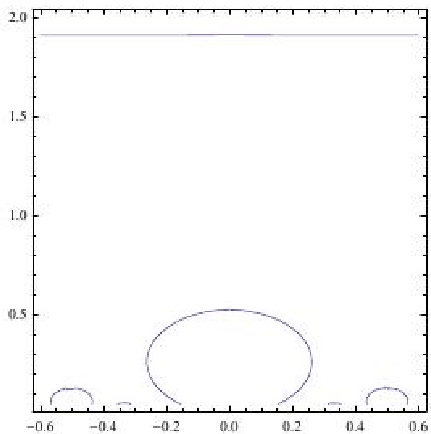
Then

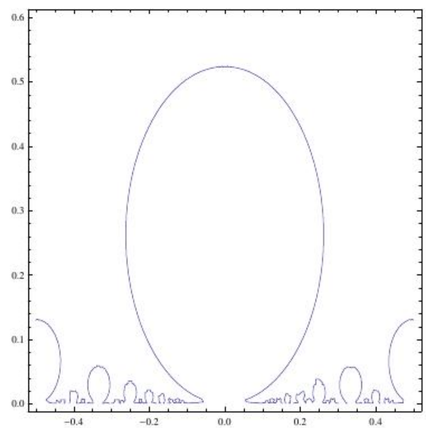
$$h(\gamma z_0) = \gamma h(z_0) = \gamma\infty = \frac{a}{c}.$$

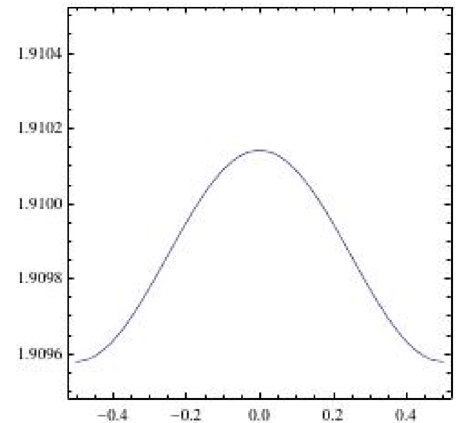
Conversely, suppose $h(\tau_0) = \frac{a}{c}$. Then
$h(\gamma^{-1}\tau_0) = \gamma^{-1}h(\tau_0) = \gamma^{-1}\frac{a}{c} = \infty$, so $E_2(\gamma^{-1}\tau_0) = 0$.

Graphs of $Im(h(z)) = 0$

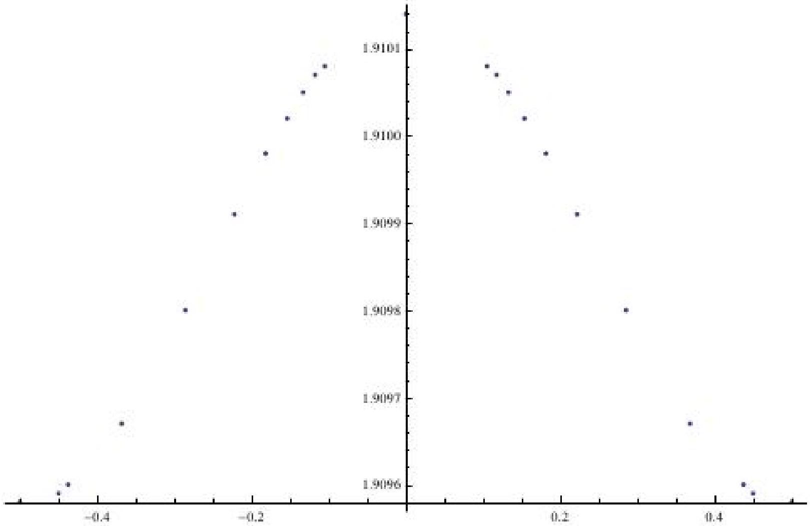
Since $h(z)$ is rational only when $E_2(z) = 0$, by graphing $\mathrm{Im}(h(z)) = 0$, all of the solutions to $E_2(z) = 0$ will be plotted along with some other values.



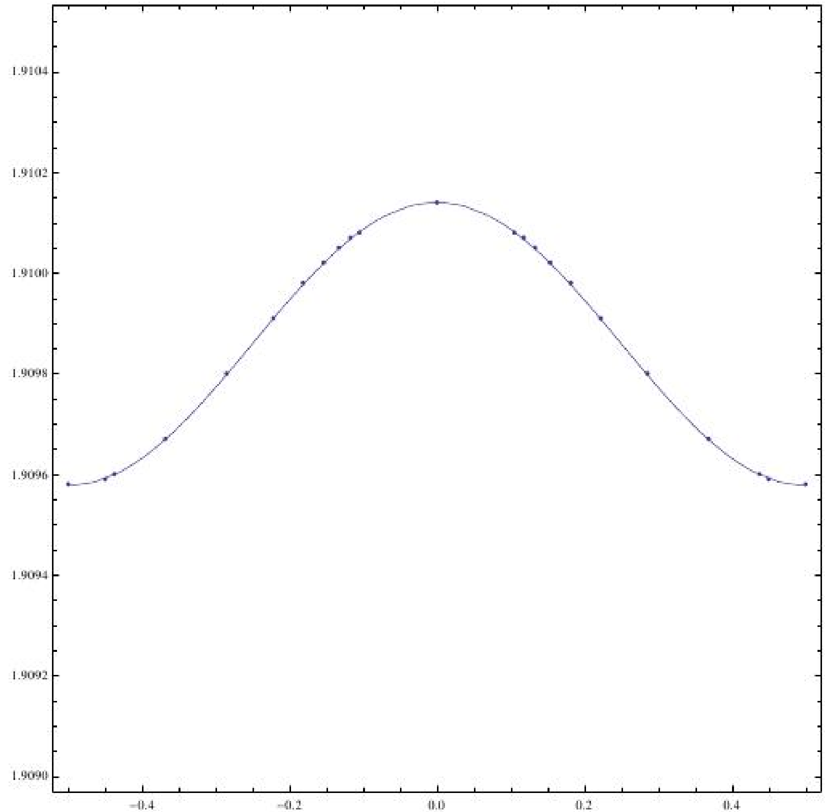




What would happen if we transformed the zeros of E_2 into $\mathbb{D}$?



Graph of
$Im(h(z)) = 0$ and (some of) the translated zeros of E_2.



When is $Im(h(z)) = 0$ in $\mathbb{D}$?

Theorem 2: The real values of the function $h(z)$ which occur in the fundamental domain D occur only in the small strip $|y - 6/\pi| < .00028$.

Results

- Our initial conjecture that values of $Re(E_2(z) = 0)$ are rational numbers was incorrect. However, we do know that values of $Re(E_2(z) = 0)$ are very close to rational numbers with small denominators.
- The curve in the $\mathbb{D}$ where $Im(h(z)) = 0$ is the generating curve for all the "almost-circles" in $\mathbb{H}$ under $SL_2(\mathbb{Z})$.
- When the zeros of $E_2(z)$ are translated into $\mathbb{D}$, they lie on the curve where $Im(h(z)) = 0$
- The real values of the function $h(z)$ which occur in the fundamental domain D occur only in the small strip $|y - 6/\pi| < .00028$.
- Lots of great experience and exposure to different areas of mathematics! :)

Acknowledgements

- My wonderful mentors Dr. Matthew Young and Dr. Sheng-Chi Liu!
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