

# NODAL DOMAINS OF EISENSTEIN SERIES

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ABSTRACT. We conduct a study of the nodal domains of Eisenstein series through a combination of numerical computations and topological and analytic arguments. This work complements that of [ ] on Maass forms.

## 1. INTRODUCTION

This is an early draft of what should be a paper ready to be submitted within about a month. In particular, the figures will need to be rearranged somewhat for the sake of clarity.

The real-analytic Eisenstein series  $E(z, s) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$ . The Eisenstein Series is  $SL(2, Z)$ -invariant and convergent for  $Re(s) > 1$ . The merimorphic continuation for the Eisenstein series is  $E^*(z, s)$ , where  $E^*(z, s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z, s)$ . The zero sets of  $E^*(z, s)$  are continuous curves that split the graph into nodal domains, areas of the curve that are entirely positive or negative.

Both Maass forms and the Eisenstein series are eigenfuntions of the hyperbolic Laplacian  $\Delta = -y^2(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2})$ . Of particular interest is the Eisenstein series with  $s = \frac{1}{2} + iT$ . Using a process analogous to Fourier analysis on periodic functions on  $\mathbb{R}/\mathbb{Z}$ , the Eisenstein series, along with the Maass waveforms, make up the building blocks for  $SL_2(\mathbb{Z})$ -invariant analytic functions that are  $L_2$  integrable. The Maass forms provide the discrete spectrum and the Eisenstein series provide the continuous spectrum.

The Maass forms and the Eisenstein series play similar, though distinct roles. In [GRS], the nodal domains of Maass forms were analyzed extensively. In this paper, we address some similar questions, as well as new questions that emerge from considering the Eisenstein series as a continuous function of  $T$ .

One immediate result that shows the differences between Maass forms and the Eisenstein series is the following:

**Theorem 1.1.** *The Eisenstein series has infinitely many nodal domains for all  $T$ .*

We begin with the proof of this result and follow with further conjectures and arguments about the properties of these nodal domains.

## 2. INFINITELY MANY NODAL DOMAINS

We have, as in, e.g., Iwaniec-Kowalski, that:

$$E^*(z, s) = \theta(s)y^s + \theta(1-s)y^{1-s} + 4\sqrt{y} \sum_{n=1}^{\infty} \tau_{s-1/2}(n)K_{s-1/2}(2\pi ny) \cos(2\pi nx),$$

where  $\theta(s) = \pi^{-s}\Gamma(2s)\zeta(s)$ .

Evaluating at the point  $s = 1/2 + iT$ , this becomes:

$$E^*(z, s) = 2\operatorname{Re}(\theta(1/2 + iT)y^{1/2+iT}) + 4\sqrt{y} \sum_{n=1}^{\infty} \tau_{iT}(n) K_{iT}(2\pi ny) \cos(2\pi nx)$$

Now, we can write  $\theta(1/2 + iT) = |\theta(1/2 + iT)|e^{i\kappa(T)}$ , where  $\kappa(T)$  is a real-valued function of  $T$ . Then, normalizing by dividing by  $|\theta(1/2 + iT)|$  and looking at the first (“constant”) term of the Eisenstein series, we get:

$$\begin{aligned} & 2\sqrt{y}\operatorname{Re}(e^{i\kappa(T)+iT\log y}) \\ &= 2\sqrt{y} \cos(\kappa(T) + T \log(y)) \end{aligned}$$

Which is straightforward to work with. (Indeed, after dividing by  $\sqrt{y}$ , and treating the expression as a function of  $\log(y)$ , it is periodic of period  $2\pi$ ).

Now, we turn our consideration to the rest of the terms in the Fourier expansion. We seek to get upper bounds on  $\frac{|\tau_{iT}(n)K_{iT}(2\pi ny)\cos(2\pi nx)|}{|\theta(1/2+iT)|}$ .

Certainly, we have  $|\cos(2\pi nx)| \leq 1$ . Also,  $|\tau_{iT}(n)| = |\sum_{ab=n}(a/b)^{iT}| \leq \sum_{d|n} 1 = d(n)$ , because  $|(a/b)^{iT}| = 1$  always. So we have the easy bound:

$$|\tau_{iT}| \leq 2\sqrt{n}$$

Moreover, we have (using the equation between 22.49 and 22.50 in [IK]) that

$$K_{iT}(2\pi ny) = \left(\frac{1}{4ny}\right)^{1/2} e^{-2\pi ny} \left(1 + \frac{\theta}{4\pi ny}\right)$$

with  $|\theta| \leq |(iT)^2 - 1/4| = 1/4 + T^2$ . Then choosing  $y > 1/(4\pi)(1/4 + T^2)$ , we have that

$$|K_{iT}(2\pi ny)| \leq 1/2(ny)^{-1/2} e^{-2\pi ny} (2) = (ny)^{-1/2} e^{-2\pi ny}$$

and combining this with our other bounds, we get that

$$|\tau_{iT}(n)K_{iT}(2\pi ny)\cos(2\pi nx)| \leq \frac{3}{\sqrt{y}} e^{-2\pi ny}.$$

So, our Eisenstein series becomes, after normalization:

$$= 2\sqrt{y} \cos(\kappa(T) + T \log(y)) + \frac{12}{|\theta(1/2 + iT)|} \sum_{n=1}^{\infty} f(n, y, T)$$

where  $|f(n, y, T)| \leq e^{-2\pi ny}$ . But the sum of these upper bounds is just a geometric series, so that the absolute value of the sum of all of the non-constant terms of the Fourier expansion is bounded above by:

$$\frac{12}{|\theta(1/2 + iT)|} \cdot \frac{e^{-2\pi y}}{1 - e^{-2\pi y}}$$

Which certainly goes to zero as  $y$  goes to infinity. So, choosing  $y$  sufficiently large, we are guaranteed that  $2 \cos(\kappa(T) + T \log(y)) + \frac{12}{|\theta(1/2+iT)|} \sum_{n=1}^{\infty} f(n, y, T)$  will be positive with  $T \log y + \kappa(T) = 2m\pi$ , and negative with  $T \log y + \kappa(T) = (2m + 1)\pi$ , for all  $m$  sufficiently large. This guarantees the existence of infinitely many nodal domains.

### 3. NODAL LINES BECOME HORIZONTAL

**Conjecture 3.1.** *Nodal lines become horizontal at around  $y = \frac{T}{2\pi}$ .*

Since  $\Gamma(2s)$  includes an implicit *etohesomething* term, the K-Bessel function reaches a local maximum at around  $y = \frac{T}{2\pi}$ . The K-Bessel function experiences rapid decay for all subsequent  $y$ , and the values of the Eisenstein series is dominated by the first term of the Fourier Expansion,

$$2\sqrt{y} \cos(\kappa(T) + T \log(y)).$$

The above expression is  $x$ -independent, and is represented graphically by horizontal lines.

### 4. SINGULARITIES

Take, as in [GRS],  $\mathbb{X} = \Gamma/\mathbb{H}$  to be the fundamental domain,  $\sigma(z) = -\bar{z}$  and  $\delta$  to be the set of points on  $\mathbb{X}$  fixed by  $\sigma$ . For each  $T > 0$ , let  $Z_T$  be the nodal set  $(z, \frac{1}{2} + iT) = 0$ . We define singularities of  $Z_T$  on  $\mathbb{X}$  in the standard way, as any point where the order of vanishing of  $E(z, \frac{1}{2} + iT)$  at a point  $z$  divided by  $Stab_{PSL(\mathbb{Z})}(z)$  is greater than 1. We define the order of a singularity in terms of dividing by the stabilizer.

**Conjecture 4.1.** *For any  $T > 0$ , there is at most one  $z \in \mathbb{X}$  (up to symmetry over  $\sigma$ ) where a singularity occurs, and if a singularity occurs it has order 2. Moreover, the sequence of  $T$ 's where singularities occur is discrete.*

### 5. COUNTING NODAL DOMAINS, ASSUMING NO SINGULARITIES

By these assumptions it follows that, except for countably many separated values of  $T$  the nodal set  $Z_T$  is a union of (countably many) disjoint simple closed curves on the fundamental domain  $\mathbb{X}$ . Call these curves  $C_1, C_2, \dots$

We can classify these curves  $C_i$  based on the number of intersections they have with  $\delta$ , the boundary of  $F_+$ . If  $T$  is such that there are no singularities on the  $C_i$ 's, and  $C_i$  crosses  $\delta$  at least twice, then taking the connected segment of the curve between some two of these intersection points gives a curve lying entirely in either  $F_+ \cup \delta$  or  $F_- \cup \delta$ . Applying  $\sigma$  gives a simple closed curve, so that in fact  $|C_i \cap \delta| \leq 2$ . Without singularities  $|C_i \cap \delta| = 1$  is impossible, so  $|C_i \cap \delta| = 0$  and  $|C_i \cap \delta| = 2$  are the only possibilities.

It is also straightforward to see (e.g. by induction) that the number of  $i$  such that  $|C_i \cap \delta| = 0$  is equal to the number of split nodal domains.

**Conjecture 5.1.** *The number of inert nodal domains is  $\lceil \frac{x}{2} \rceil + 1 + \theta$ , where  $\theta = \{0, 1, -1\}$  and  $x$  is the number of zeroes on  $\delta$ .*

*Proof.* In the absence of singularities, nodal curves that intersect  $\delta$  exclusively contribute to the number of inert nodal domains and such curves must intersect  $\delta$  exactly twice, and without intersecting any other nodal curves. Each curve introduces another inert, connected component into the graph. Thus the number of inert nodal domains is  $\frac{x}{2} + 1$ , where the extra one accounts for the original inert nodal domain that we start with before we add in the curves – the entire fundamental domain.  $\square$

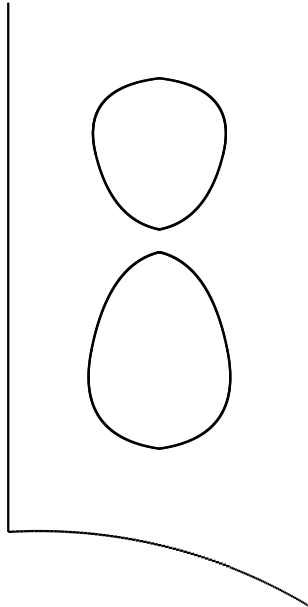
### 6. CASE STUDY WITH SINGULARITIES

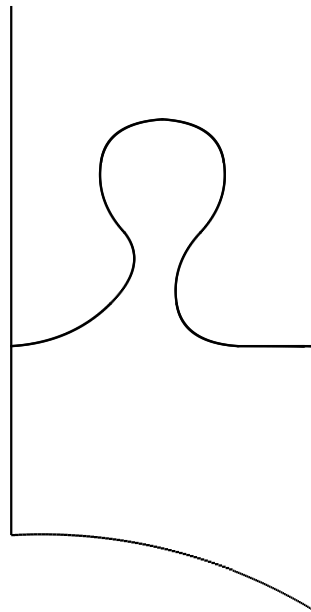
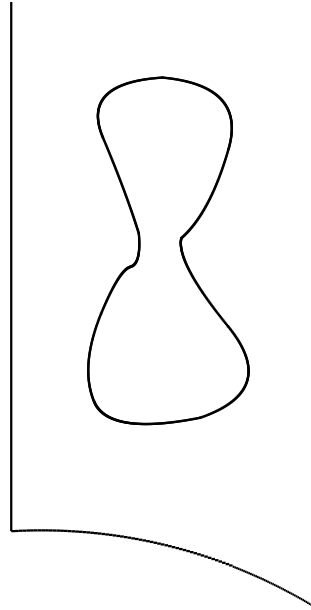
Now we shall examine how singularities alter these counts.

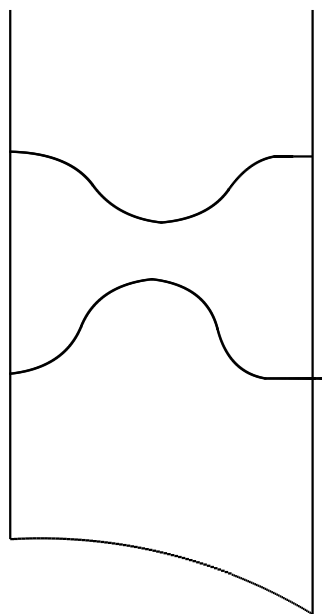
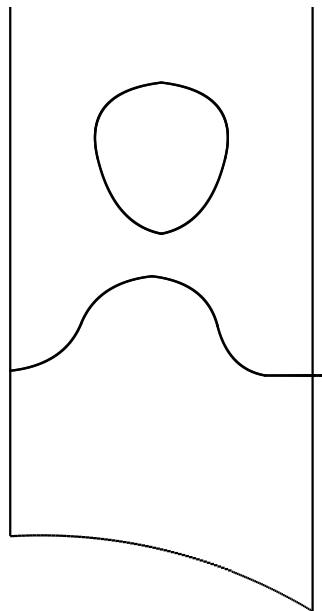
**6.1. Conjecture 5.1, continued.** Since we assume at most one singularity, and all singularities are of order two, there can be intersections of at most two inert nodal curves. In such a case, each intersection counts for one inert nodal domains, bringing our inert count, so  $\theta = 1$ . In the case where we have a singularity on  $\delta$  and all ends of the nodal curve touch the same point on  $\delta$ , then we will be creating 2 new split nodal domains, and decreasing the inert count by 1, so  $\theta = -1$ . Hence, where singularities exist, the appropriate value of  $\theta$  will adjust the count.

**6.2. Counting nodal domains where singularities exist.** First, consider the case of a singularity of order 2 at some point  $z \notin \delta$ . At the singularity, we locally appear to have four real-analytic curves  $A, B, C,$  and  $D$  which all meet at  $z$ . Tracing each of these curves (away from  $z$ ) they either approach  $z$  again or approach  $\sigma(z)$ . For example, the curve  $A$  is either connected to the curve  $B$ , the curve  $D$ , or to  $\sigma(z)$  ( $A$  cannot be connected to  $C$ . Why?). Note that  $A$  crosses  $\delta$  if and only if  $A$  crosses both  $z$  and  $\sigma(z)$ .

Looking at the nodal sets  $Z_{T-\varepsilon}$  and  $Z_{T+\varepsilon}$ , for sufficiently small  $\varepsilon > 0$ , it follows that there must be 0, 2, or 4 total intersections of  $\delta$  with  $A \cup B \cup C \cup D$ , which give rise to one of the following five figures:



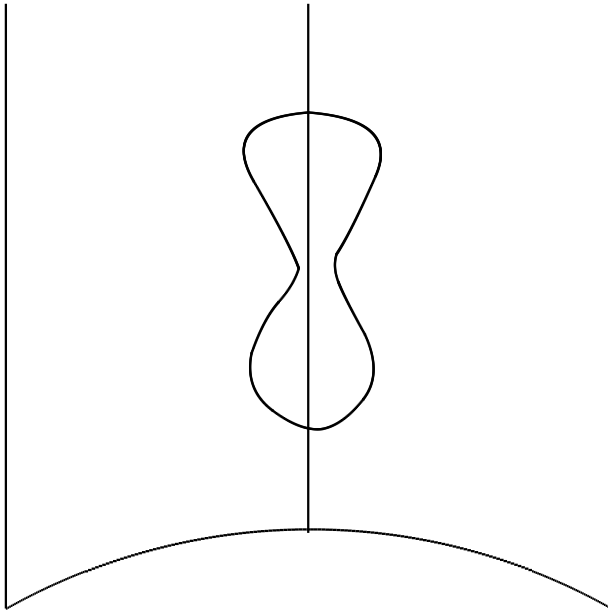
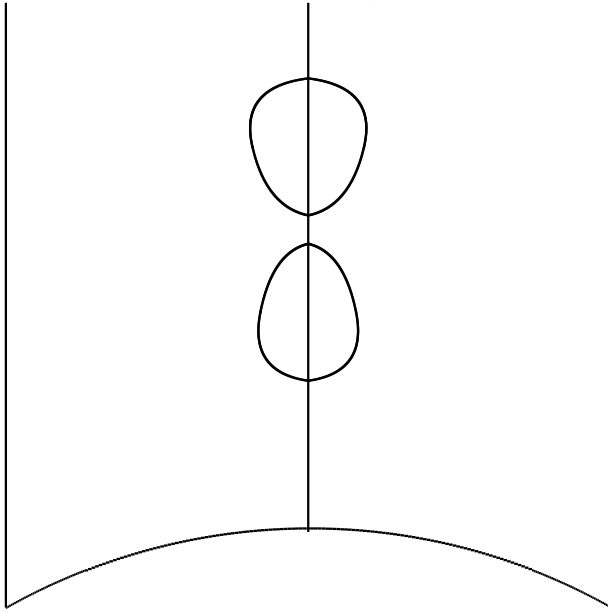


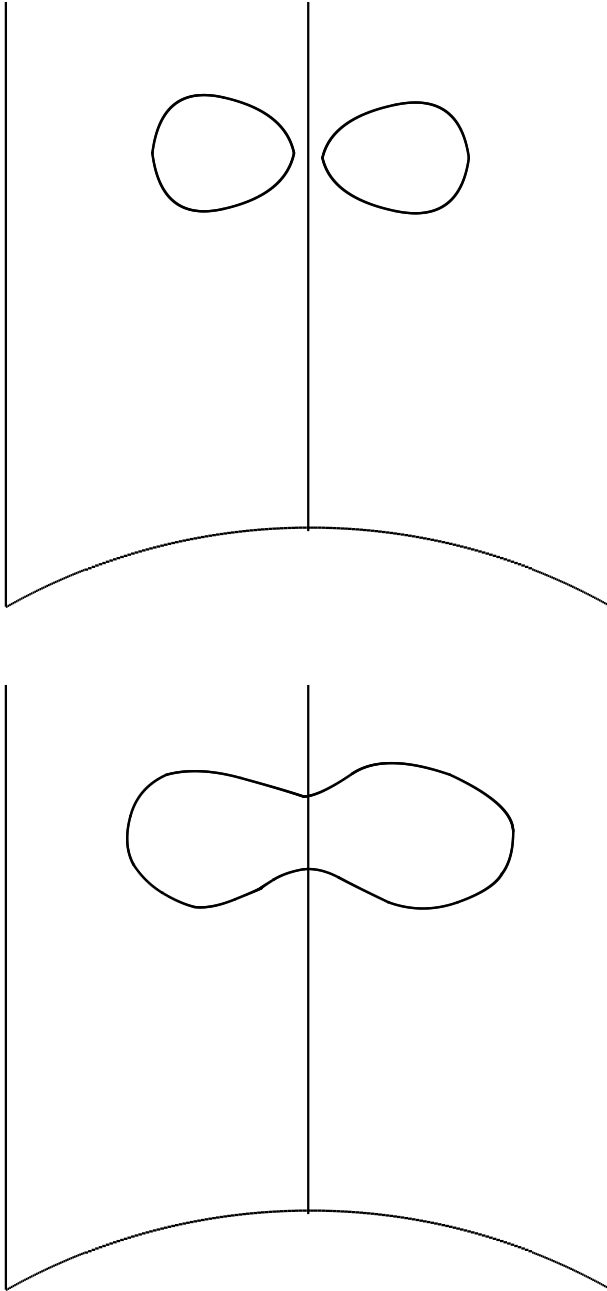


Moving to  $T + \varepsilon$  will permute the first two cases and the second two cases, leaving the fifth case fixed.

We can summarize these results as follows: If a nodal domain is self-singular on  $F_+$  (i.e. at  $T - \varepsilon$ , the four branches  $A, B, C$ , and  $D$  all belong to the same closed curve), then the number of split nodal domains increases by one. If a nodal domain intersects a different split nodal domain on  $F_+$ , the number of split nodal domains decreases by one. Finally, if a singularity on  $F_+$  is formed by the intersection of two distinct inert nodal domains on  $F_+$ , no change occurs in the number of nodal domains.

We now consider what occurs if at  $T$  there is a singularity at some  $z \in \delta$ . We have again our branches  $A, B, C$ , and  $D$ , emanating from  $z$ . Also, again, there are either 0, 2, or 4 intersections with  $\delta$  at  $T - \varepsilon$ , which leads to the following four cases:





Moving past the singularity to  $T + \varepsilon$  permutes the first two and the last two figures.

We can summarize these results as follows: If a split nodal domain encounters its image under  $\sigma$  at  $\delta$ , then the number of inert nodal domains increases by one, and the number of split nodal domains decreases by two. If two inert nodal domains intersect at  $\delta$ , then the number of inert nodal domains decreases by one. Finally, if an inert nodal domain encounters itself at  $\delta$ , then either the number of inert nodal domains increases by one (Fig ??) or the number of inert nodal domains decreases by one and the number of split nodal domains increases by two (Fig ??).



## 7. INERT NODAL DOMAINS

Using our analysis on local singularities and the ways in which they affect the number of nodal domains as well as numerical data, we have the following conjecture on the number of inert nodal domains for any fixed compact set.

**Conjecture 7.1.**

$$N_i \approx \frac{T \log\left(\frac{y_2}{y_1}\right)}{\pi}$$

where  $y_2$  and  $y_1$  represent the range of the compact set and  $y_2 > y_1$ . In other words,  $N_i$  is linear with respect to  $T$ .

The first term of the Fourier expansion gives rise to horizontal lines, which are inert nodal domains. We can interpret the remaining terms with the K-Bessel functions to play a role in deforming these horizontal lines and creating singularities that cause fluctuations in the number of nodal domains.

We isolate our analysis of singularities to  $\delta$ , since singularities on the interior do not affect the number of inert nodal domains. If a certain configuration of singularity creates a new inert nodal domain, the inverse will destroy an inert nodal domain. Numerical analysis suggests that both are equally likely to happen. Therefore, the number of inert nodal domains can be approximated by the "source", which are the horizontal lines of the constant term.

## 8. CONCLUSION

We are looking further into the statistical properties of the zeroes of the Eisenstein function. This conclusion will be fleshed out in the final draft of the paper with observations combining our intuition built up from numerical data and reading the relevant literature.